Finite Section Method for a Banach Algebra of Convolution Type Operators on $L^p(\mathbb{R})$ with Symbols Generated by PC and SO

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Abstract. We prove the applicability of the finite section method to an arbitrary operator in the Banach algebra generated by the operators of multiplication by piecewise continuous functions and the convolution operators with symbols in the algebra generated by piecewise continuous and slowly oscillating Fourier multipliers on $L^p(\mathbb{R})$, $1 < p < \infty$.

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1. Introduction

Given $1 < p < \infty$, let $\mathcal{B} := \mathcal{B}(L^p(\mathbb{R}))$ denote the Banach algebra of all bounded linear operators on the Lebesgue space $L^p(\mathbb{R})$. Let $[PC, SO]$ be the smallest C^*-subalgebra of $L^\infty(\mathbb{R})$ containing all piecewise continuous (PC) and slowly oscillating (SO) functions, and let $[PC_p, SO_p]$ stand for its Fourier multiplier analogue, which is a Banach subalgebra of $\mathcal{M}_p$, the Banach algebra of all Fourier multipliers on $L^p(\mathbb{R})$. The Fredholm theory for the smallest Banach subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$ that contains all the convolution type operators $aF^{-1}bF$ where F is the Fourier transform given by

$$(F\varphi)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \varphi(y)\, dy \quad (x \in \mathbb{R}) \tag{1.1}$$

and $a \in [PC, SO]$, $b \in [PC_p, SO_p]$ is constructed in [1, 2] (in those papers, even matrix functions a and b are considered).

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Let $\mathbb{R}_+ := (0,\infty)$ and $\mathbb{R}_- := (-\infty,0)$. For $\tau \in \mathbb{R}_+$, consider the operators

$$(P_\tau\varphi)(t) := \begin{cases} \varphi(t) & \text{if} \quad |t| < \tau, \\ 0 & \text{if} \quad |t| > \tau, \end{cases} \qquad Q_\tau := I - P_\tau$$

acting on $L^p(\mathbb{R})$ with norm 1. Clearly, $P_\tau \to I$ and $Q_\tau \to 0$ strongly as $\tau \to \infty$. One says that *the finite section method applies* to an operator $A \in \mathcal{B}(L^p(\mathbb{R}))$ if there exists a positive constant τ_0, such that for any $\tau > \tau_0$ and any $f \in L^p(\mathbb{R})$ there exists a unique solution φ_τ of the equation

$$A_\tau\varphi_\tau := (P_\tau A P_\tau + Q_\tau)\varphi_\tau = f \quad (\tau \in \mathbb{R}_+) \tag{1.2}$$

and φ_τ converges in the norm of $L^p(\mathbb{R})$ to a solution of the equation $A\varphi = f$ as $\tau \to \infty$.

We refer to the monographs by Gohberg and Feldman [7], Prössdorf and Silbermann [16], Hagen, Roch, and Silbermann [10, 11] for a general theory of projection methods as well as for more specific issues of the finite section method for convolution type operators and algebras generated by them.

Roch, Silbermann, and one of the authors [18] studied the applicability of the finite section method to an operator in the smallest C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}))$ generated by all aI with $a \in PC$, $W^0(b)$ with $b \in PC = PC_2$, and the so-called flip operator. For general $p \neq 2$, they proved in [20] the applicability of the finite section method for an arbitrary operator in the Banach algebra generated by the operators of multiplication by piecewise continuous functions (PC) and by the convolution operators with piecewise continuous Fourier multipliers (PC_p).

The aim of this paper is to take one more step forward. We prove the applicability of the finite section for an arbitrary operator A in the Banach subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$, $1 < p < \infty$, generated by all operators aI of multiplication by functions $a \in PC$ and by all Fourier convolution operators

$$W^0(b) := F^{-1}bF$$

with $b \in [PC_p, SO_p]$.

Our approach to analyze the applicability of the finite sections method will follow a general scheme to treat approximation problems. This scheme goes back to Silbermann [23] (see also [10, Section 1.6]). It can be summarized as follows. Let $\mathcal{A}$ be a set of (generalized) sequences that contains all sequences of the form

$$(A_\tau) = (P_\tau A P_\tau + Q_\tau) \quad (\tau \in \mathbb{R}_+). \tag{1.3}$$

1. **Algebraization**: Find a unital Banach algebra $\mathcal{E}$ containing $\mathcal{A}$ and a closed ideal $\mathcal{G}$ of $\mathcal{E}$ such that the original problem becomes equivalent to an invertibility problem in the quotient algebra $\mathcal{E}/\mathcal{G}$.
2. **Essentialization**: Find a unital inverse closed subalgebra $\mathcal{F}$ of $\mathcal{E}$ that contains $\mathcal{A}$ and a closed ideal $\mathcal{J}$ of $\mathcal{F}$ that contains $\mathcal{G}$, such that $\mathcal{J}$ can be lifted. The latter means that one has full control about the difference between the invertibility of a coset of a sequence $(A_\tau) \in \mathcal{F}$ in the algebra $\mathcal{F}/\mathcal{G}$ and the invertibility of the coset of the same sequence in $\mathcal{F}/\mathcal{J}$. This control is usually guaranteed by a lifting theorem.

3. **Localization**: Find a unital subalgebra $\mathcal{L}$ of $\mathcal{F}$ such that
   (a) $\mathcal{A}, \mathcal{J} \subset \mathcal{L}$;
   (b) $\mathcal{L}/\mathcal{J}$ is inverse closed in $\mathcal{F}/\mathcal{J}$;
   (c) the quotient algebra $\mathcal{L}/\mathcal{J}$ has a large center.
   Use a local principle to translate the invertibility problem in the algebra $\mathcal{L}/\mathcal{J}$ to a family of simpler invertibility problems in the local algebras.
4. **Identification**: Find conditions for the invertibility of the cosets of sequences in $\mathcal{A}$ in the local algebras.

The paper is organized as follows. Section 2 contains all necessary properties of piecewise continuous and slowly oscillating Fourier multipliers. In Section 3, we formulate several auxiliary results on singular integral operators with piecewise constant coefficients, which will be used in Sections 10 and 11. In Section 4, the algebra $\mathcal{E}$ and its ideal $\mathcal{G}$ are defined and Kozak's theorem is formulated.

Let $\mathcal{A}$ denote the smallest closed subalgebra of $\mathcal{E}$ that contains the constant sequences (aI) with $a \in PC$ and $(W^0(b))$ with $b \in [PC_p, SO_p]$ and the sequence (P_τ). In Section 5, we perform the essentialization step: we introduce the algebra $\mathcal{F} \subset \mathcal{E}$, its ideal $\mathcal{J}$, and the homomorphisms W_i, $i \in \{-1, 0, 1\}$. It is shown that $\mathcal{A} \subset \mathcal{F}$. The main result of Section 5 (Theorem 5.5) says that for a sequence $\mathbf{A} = (A_\tau) \in \mathcal{F}$ the coset $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$ if and only if the operators $\mathsf{W}_i(\mathbf{A})$, $i \in \{-1, 0, 1\}$, are invertible and the coset $\mathbf{A} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$. The algebra $\mathcal{F}/\mathcal{J}$ is still too large for effective studying.

In Section 6, we introduce the algebra $\mathcal{L}$ of sequences of local type such that $\mathcal{A} \subset \mathcal{L} \subset \mathcal{F}$ and the algebra $\mathcal{L}/\mathcal{J}$ has a large center. So the latter algebra can be studied with the aid of the Allan-Douglas local principle. According to it, the invertibility of $\mathbf{A} + \mathcal{J} \in \mathcal{L}/\mathcal{J}$ is equivalent to the invertibility of the local representatives $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A})$ in the local algebras $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$, where

$$(\xi, \eta) \in \big(\mathbb{R} \times M_\infty(SO)\big) \cup \big(\{\infty\} \times \mathbb{R}\big) \cup \big(\{\infty\} \times M_\infty(SO)\big)$$

and $M_\infty(SO)$ is the fiber of the maximal ideal space of SO over the point ∞ (see Section 2.3).

In Section 7, we introduce the homomorphisms H_η for $\eta \in \mathbb{R}$ and show that if a sequence $\mathbf{A} \in \mathcal{A}$ is stable, then all operators $\mathsf{H}_\eta(\mathbf{A})$ for $\eta \in \mathbb{R}$ are invertible.

In Sections 8–10, we study the invertibility in the local algebras $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$. It turns out that these algebras are too large for a complete description, so we will restrict ourselves to studying the invertibility in their subalgebras $\mathcal{A}^{\mathcal{J}}_{\xi,\eta} := \Phi^{\mathcal{J}}_{\xi,\eta}(\mathcal{A})$.

Let $\mathbf{A} \in \mathcal{A}$. In Section 8, we prove that the invertibility of the operator $\mathsf{W}_0(\mathbf{A})$ is sufficient for the invertibility of $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A})$ in the local algebra $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$ for $(\xi, \eta) \in \mathbb{R} \times M_\infty(SO)$. Further, in Section 9, we obtain that the invertibility of the operator $\mathsf{H}_\eta(\mathbf{A})$ is sufficient for the invertibility of $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})$ in the local algebra $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ with $\eta \in \mathbb{R}$.

Section 10 is devoted to $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$ and $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ with $\eta \in M_\infty(SO)$. We show that the invertibility problem in these algebras can be reduced to the invertibility problem in the pairs of simpler algebras $\mathcal{A}^{\pm}_\eta$ and $\mathcal{L}^{\pm}_\eta$. It turns out that the algebras $\mathcal{A}^{\pm}_\eta$ are

generated by two idempotents and the identities of the algebras $\mathcal{L}_\eta^\pm$. Applying the two-idempotents theorem (Theorem 10.5), we get necessary and sufficient conditions for the invertibility of an element of $\mathcal{A}_\eta^\pm$ in the algebra $\mathcal{L}_\eta^\pm$. These results lead to a criterion of the invertibility of an element of $\mathcal{A}_{\infty,\eta}^{\mathcal{J}}$ in the algebra $\mathcal{L}_{\infty,\eta}^{\mathcal{J}}$.

In Section 11 we gather the results obtained in Sections 6–10 and prove our main result: a criterion for the stability of a sequence $\mathbf{A} = (A_\tau) \in \mathcal{A}$. Further, it is specified for the sequences of the finite section method $(P_\tau A P_\tau + Q_\tau)$. Finally we illustrate our results for the paired convolution operators

$$A = W^0(a)\chi_- I + W^0(b)\chi_+ I$$

with $a, b \in [PC_p, SO_p]$, where χ_- and χ_+ are the characteristic functions of the half-axes $(-\infty, 0)$ and $(0, +\infty)$.

2. Piecewise continuous and slowly oscillating Fourier multipliers

2.1. Function algebras

Let $1 \le p \le \infty$ and $L^p(\mathbb{R})$ denote the usual Lebesgue space on $\mathbb{R} = (-\infty, +\infty)$ with the standard norm denoted by $\|\cdot\|_p$, let $C(\overline{\mathbb{R}})$ and $C(\dot{\mathbb{R}})$ denote the spaces of continuous functions on $\overline{\mathbb{R}} = [-\infty, +\infty]$ and $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, respectively;

$$C_0(\mathbb{R}) := \big\{ f \in C(\dot{\mathbb{R}}) \ : \ f(\infty) = 0 \big\}, \quad C_b(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R});$$

and PC stand for the set of all functions $f : \dot{\mathbb{R}} \to \mathbb{C}$ which possess a finite left-hand limit $f(x^-)$ and a finite right-hand limit $f(x^+)$ at every point $x \in \dot{\mathbb{R}}$.

Let $V_1(\mathbb{R})$ be the set of functions $a : \overline{\mathbb{R}} \to \mathbb{C}$ with the finite total variation

$$V_1(a) := \sup \left\{ \sum_{i=1}^{n} |a(x_i) - a(x_{i-1})| : \ -\infty \le x_0 < x_1 < \dots < x_n \le +\infty, n \in \mathbb{N} \right\}.$$

where the supremum is taken over all finite decompositions of the real line $\mathbb{R}$. It is well known that $V_1(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{V_1(\mathbb{R})} := \|a\|_\infty + V_1(a).$$

For a continuous function $f : \mathbb{R} \to \mathbb{C}$ and a set $I \subset \mathbb{R}$, let

$$\operatorname{osc}(f, I) := \sup_{t,s \in I} |f(t) - f(s)|.$$

Following Power [15] we denote by SO the set of all *slowly oscillating functions*,

$$SO := \left\{ f \in C_b(\mathbb{R}) \ : \ \lim_{x \to +\infty} \operatorname{osc}(f, [-2x, -x] \cup [x, 2x]) = 0 \right\}.$$

Clearly, SO is a C^*-subalgebra of $L^\infty(\mathbb{R})$ and $C(\dot{\mathbb{R}}) \subset SO$.

Let $C_b^1(\mathbb{R})$ consist of all functions $a \in C_b(\mathbb{R})$ with $a' \in C_b(\mathbb{R})$. The following lemma is, in fact, proved in [1, pp. 154–155].

Lemma 2.1. *If a function a is even and lies in*

$$\widetilde{SO}^1 := \left\{ f \in C_b^1(\mathbb{R}) : \lim_{x\to\infty} x f'(x) = 0 \right\},$$

then $a \in SO$.

However, $\widetilde{SO}^1$ is not contained in SO because its elements can have different slowly oscillating behavior at $-\infty$ and $+\infty$.

Lemma 2.2 ([1, Corollary 2.4]). *Every function $a \in SO$ can be uniformly approximated by functions in the non-closed algebra*

$$SO^1 := \widetilde{SO}^1 \cap SO. \tag{2.1}$$

From now on, we consider $1 < p < \infty$. Let $\mathcal{B} := \mathcal{B}(L^p(\mathbb{R}))$ be the Banach algebra of all bounded linear operators on $L^p(\mathbb{R})$ and $\mathcal{K} := \mathcal{K}(L^p(\mathbb{R}))$ be the closed two-sided ideal of all compact operators on $L^p(\mathbb{R})$. The Cauchy singular integral operator $S_\mathbb{R}$ is defined on $L^p(\mathbb{R})$ by

$$(S_\mathbb{R} f)(x) := \frac{1}{\pi i} \int_\mathbb{R} \frac{f(t)}{t-x}\, dt \quad (x \in \mathbb{R})$$

where the integral is understood in the sense of principal value.

Let $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform given by (1.1), and let F^{-1} be its inverse. We define the operator $W^0(a)$ on $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ by

$$(W^0(a)\varphi)(x) := (F^{-1} a F \varphi)(x), \quad \varphi \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}). \tag{2.2}$$

A function $a \in L^\infty(\mathbb{R})$ is called a *Fourier multiplier on* $L^p(\mathbb{R})$ if the operator $W^0(a)$ given by (2.2) can be extended to a bounded linear operator on $L^p(\mathbb{R})$, which will again be denoted by $W^0(a)$. By χ_+ (resp. χ_-) denote the characteristic function of the semi-axis $\mathbb{R}_+ := (0, +\infty)$ (resp. $\mathbb{R}_- := (-\infty, 0)$). Then

$$W^0(\chi_-) = P_\mathbb{R} := (I + S_\mathbb{R})/2, \quad W^0(\chi_+) = Q_\mathbb{R} := (I - S_\mathbb{R})/2$$

are two complementary projections on $L^p(\mathbb{R})$ (see e.g. [6, Section 2] or [4, Section 2.5]).

The set $\mathcal{M}_p$ of all Fourier multipliers on $L^p(\mathbb{R})$ is defined as

$$\mathcal{M}_p := \left\{ a \in L^\infty(\mathbb{R}) \ : \ W^0(a) \in \mathcal{B}(L^p(\mathbb{R})) \right\}.$$

It is well known that $\mathcal{M}_p$ is a Banach algebra with the norm

$$\|a\|_{\mathcal{M}_p} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R}))},$$

and

$$\mathcal{M}_p \subset \mathcal{M}_2 = L^\infty(\mathbb{R}) \quad \text{for all} \quad p \in (1, \infty). \tag{2.3}$$

According to Stechkin's inequality (see e.g. [4, Theorem 17.1]), $\mathcal{M}_p$ contains all functions $a \in PC$ of finite total variation and

$$\|a\|_{\mathcal{M}_p} \le \|S_\mathbb{R}\|_\mathcal{B} \big(\|a\|_\infty + V_1(a) \big).$$

Let $C_p(\dot{\mathbb{R}})$ and $C_p(\overline{\mathbb{R}})$ stand for the closure in $\mathcal{M}_p$ of the set of all functions with finite total variation in $C(\dot{\mathbb{R}})$ and $C(\overline{\mathbb{R}})$, respectively. Denote by PC_p the closure

in $\mathcal{M}_p$ of the set of all piecewise constant functions on $\mathbb{R}$ which have at most finite sets of jumps.

Lemma 2.3 ([24, Lemma 1.1])**.** *If $1 < p < \infty$, then the algebra $C_p(\overline{\mathbb{R}})$ is generated by the functions $f(x) = 1$ and $g(x) = \tanh x$.*

In view of (2.1), every function $a \in SO^1$ has the properties $a \in C_b(\mathbb{R})$ and $Da \in C_b(\mathbb{R})$ where $(Da)(x) = xa'(x)$. Therefore, by Mikhlin's theorem (see e.g. [9, Theorem 5.2.7(a)]), every function $a \in SO^1$ belongs to all spaces $\mathcal{M}_p$ for $p \in (1, \infty)$. Hence, by analogy with $C_p(\dot{\mathbb{R}})$ and taking into account Lemma 2.2, one can define the set SO_p of slowly oscillating Fourier multipliers as the closure in $\mathcal{M}_p$ of the set SO^1. Clearly, SO_p is a Banach subalgebra of $\mathcal{M}_p$. The set SO_p was introduced in [1, 2]. From (2.3) it follows that

$$SO^1 \subset SO_p \subset SO_2 = SO \quad \text{for all} \quad 1 < p < \infty. \tag{2.4}$$

For $a \in L^\infty(\mathbb{R})$ by $\overline{a}$ denote the function $\overline{a}(t) := \overline{a(t)}$, where the bar denotes the complex conjugation.

Lemma 2.4. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. If $a \in SO_p$, then $\overline{a} \in SO_q$.*

Proof. Let a be the limit of a sequence $(a_n)_{n=1}^\infty \subset SO^1$ in the norm of $\mathcal{M}_p$. Obviously, $\overline{a_n} \in SO^1 \subset SO_q$. Since $a - a_n \in \mathcal{M}_p$, from [5, Section 9.3(b)] we see that $\overline{a} - \overline{a_n} \in \mathcal{M}_q$ and $\|\overline{a} - \overline{a_n}\|_{\mathcal{M}_q} = \|a - a_n\|_{\mathcal{M}_p} \to 0$ as $n \to \infty$. This means that $\overline{a} \in SO_q$. $\square$

We denote by $[PC, SO]$ the smallest C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains PC and SO. Similarly, for every $p \in (1, \infty)$, we introduce the Banach subalgebra $[PC_p, SO_p]$ of $\mathcal{M}_p$ generated by PC_p and SO_p. Obviously,

$$[PC_p, SO_p] \subset [PC_2, SO_2] = [PC, SO] \quad \text{for all} \quad 1 < p < \infty.$$

2.2. Compactness of Hankel type operators and commutators

Put

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\},$$

and denote by $H^\infty(\mathbb{C}_\pm)$ the set of all bounded and analytic functions in $\mathbb{C}_\pm$. Fatou's theorem says that functions in $H^\infty(\mathbb{C}_\pm)$ have non-tangential limits on $\mathbb{R}$ almost everywhere, and we denote by $H^\infty_\pm$ the set of all $a \in L^\infty(\mathbb{R})$ that are non-tangential limits of functions in $H^\infty(\mathbb{C}_\pm)$. It is well known that $H^\infty_\pm$ are closed subalgebras of $L^\infty(\mathbb{R})$. Sarason discovered (see e.g. [5, p. 107]) that the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains both $C(\dot{\mathbb{R}})$ and $H^\infty_\pm$ coincides with the sum of $C(\dot{\mathbb{R}})$ and $H^\infty_\pm$.

Lemma 2.5. *Let $1 < p < \infty$. If $b \in SO_p$, then $\chi_+ W^0(b) \chi_- I$ and $\chi_- W^0(b) \chi_+ I$ are compact on $L^p(\mathbb{R})$.*

Proof. From [5, Sections 9.11, 9.35, and 2.80] it follows that

$$SO \subset (C(\dot{\mathbb{R}}) + H_+^\infty) \cap (C(\dot{\mathbb{R}}) + H_-^\infty). \tag{2.5}$$

By Hartman's theorem (see e.g. [4, Theorem 2.18] and also [5, Section 2.54]), if $b \in C(\dot{\mathbb{R}}) + H_-^\infty$ (resp. $b \in C(\dot{\mathbb{R}}) + H_+^\infty$), then the operator $\chi_+ W^0(b)\chi_- I$ (resp. $\chi_- W^0(b)\chi_+ I$) is compact on $L^2(\mathbb{R})$.

Let $b \in SO_p$ be the limit in the norm of $\mathcal{M}_p$ of a sequence $b_n \in SO^1$. From the above results and (2.4)–(2.5) it follows that $\chi_\pm W^0(b_n)\chi_\mp I$ are compact on $L^2(\mathbb{R})$. Moreover, these operators are bounded on every $L^p(\mathbb{R})$ with $p \in (1,\infty)$. By the Krasnosel'skii interpolation theorem [12] (see also [13, Theorem 3.10]), the operators $\chi_\pm W^0(b_n)\chi_\mp I$ are compact on $L^p(\mathbb{R})$ for every $p \in (1,\infty)$. From

$$\begin{aligned}\|\chi_\pm W^0(b)\chi_\mp I - \chi_\pm W^0(b_n)\chi_\mp I\|_{\mathcal{B}(L^p(\mathbb{R}))} &\le \|W^0(b-b_n)\|_{\mathcal{B}(L^p(\mathbb{R}))} \\ &= \|b-b_n\|_{\mathcal{M}_p} = o(1)\end{aligned}$$

as $n \to \infty$ it follows that the operators $\chi_\pm W^0(b)\chi_\mp I$ are compact on $L^p(\mathbb{R})$. □

Lemma 2.6 ([6, Lemma 7.4]). *Let $1 < p < \infty$. If $a \in C(\overline{\mathbb{R}})$ and $b \in C_p(\overline{\mathbb{R}})$, then the operator $aW^0(b) - W^0(b)aI$ is compact on the space $L^p(\mathbb{R})$.*

Lemma 2.7 ([1, Theorem 4.2]). *Let $1 < p < \infty$. If $a \in [PC, SO]$, $b \in SO_p$ or $a \in SO$, $b \in [PC_p, SO_p]$, then the operator $aW^0(b) - W^0(b)aI$ is compact on the space $L^p(\mathbb{R})$.*

2.3. Maximal ideal spaces of some commutative Banach algebras

Let $M(SO)$ denote the maximal ideal space of SO. Identifying points $t \in \dot{\mathbb{R}}$ with the evaluation functionals at t, one can identify the fiber of $M(SO)$ over $t \in \dot{\mathbb{R}}$ by

$$M_t(SO) = \big\{\eta \in M(SO) \;:\; \eta|_{C(\dot{\mathbb{R}})} = t\big\}.$$

If $t \in \mathbb{R}$, then the fiber $M_t(SO)$ consists of the only evaluation functional at t, and thus

$$M(SO) = \bigcup_{t\in\dot{\mathbb{R}}} M_t(SO) = \mathbb{R} \cup M_\infty(SO).$$

According to [1, Proposition 3.1], the fiber $M_\infty(SO)$ has the form

$$M_\infty(SO) = (\operatorname{clos}_{SO^*}\mathbb{R}) \setminus \mathbb{R}$$

where $\operatorname{clos}_{SO^*}\mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*, the dual space of SO. Thus, for every functional $\eta \in M_\infty(SO)$ there exists a net $t_\omega \in \mathbb{R}$ that tends to ∞ in the usual topology of $\mathbb{R}$ and such that η is the limit of t_ω in the Gelfand topology, that is, $\eta(a) = \lim_\omega a(t_\omega)$ for every $a \in SO$. The next lemma shows, in particular, that for a fixed $a \in SO$, the net t_ω can be replaced by a sequence $\tau_n \to +\infty$.

Lemma 2.8 ([1, Corollary 3.3]). *If $(a_k)_{k=1}^\infty$ is a countable subset of SO and η is an element of $M_\infty(SO)$, then there exists a sequence $(\tau_n)_{n=1}^\infty \subset \mathbb{R}_+$ such that $\tau_n > 1$, $\tau_n \to +\infty$ as $n \to \infty$, and for every $x \in \mathbb{R}\setminus\{0\}$,*

$$\eta(a_k) = \lim_{n\to\infty} a_k(\tau_n x) \quad (k \in \mathbb{N}).$$

In what follows we write

$$a(\eta) := \eta(a)$$

for every $a \in SO$ and every $\eta \in M(SO)$.

A unital Banach algebra A is called *inverse closed* in a unital Banach algebra $B \supset A$ with the same unit if for any $a \in A$ invertible in B its inverse a^{-1} belongs to A.

From [1, Section 3] it follows that the Banach algebras SO_p and $[PC_p, SO_p]$ are inverse closed in the C^*-algebras SO and $[PC, SO]$, respectively, and their maximal ideal spaces coincide as sets:

$$M(SO_p) = M(SO), \quad M([PC_p, SO_p]) = M([PC, SO]).$$

It is well known that $M_\infty(PC) = \{\pm\infty\}$. According to [15, Section 1], the fiber $M_\infty([PC, SO])$ is homeomorphic to the product

$$M_\infty(PC) \times M_\infty(SO) = \{\pm\infty\} \times M_\infty(SO)$$

and the homeomorphism is given by the restriction map $\beta \mapsto (\beta|_{PC}, \beta|_{SO})$. Thus, every $\beta \in M_\infty([PC, SO])$ can be viewed as a functional of the form either $(+\infty, \eta)$ or $(-\infty, \eta)$ with $\eta \in M_\infty(SO)$. Fix $\eta \in M_\infty(SO)$. Then there exists a homomorphism

$$\alpha_\eta : [PC, SO] \to PC|_{\{\pm\infty\}}, \quad (\alpha_\eta a)(\pm\infty) = a_\eta(\pm\infty) := (\pm\infty, \eta)a. \tag{2.6}$$

As the values $a_\eta(\pm\infty)$ are uniquely defined, for every $\eta \in M_\infty(SO)$ we get the homomorphism

$$\gamma_\eta : [PC, SO] \to PC, \quad \gamma_\eta a = a_\eta(-\infty)\chi_- + a_\eta(+\infty)\chi_+, \tag{2.7}$$

where $a_\eta(\pm\infty)$ are defined by (2.6).

As $M_\infty(PC_p) = M_\infty(PC) = \{\pm\infty\}$ for every $p \in (1, \infty)$, we infer that the restriction of the homomorphism α_η to $[PC_p, SO_p]$ sends this algebra to $PC_p|_{M_\infty(PC)} = PC|_{M_\infty(PC)}$ according to (2.6). Therefore, for all $\eta \in M_\infty(SO)$, the homomorphisms γ_η given by (2.7) map $[PC_p, SO_p]$ into PC_p.

To have an idea how γ_η acts, we give the following example.

Example. Let $k \in \mathbb{N}$ and $c_k, d_k \in \mathbb{C}$. Put

$$f_k(t) := c_k\chi_-(t) + d_k\chi_+(t), \quad g_k(t) := \frac{t^2}{t^2+1}\exp\left(i\sqrt{\log(t^{2k}+1)}\right) \quad (t \in \mathbb{R}).$$

The functions f_k are piecewise constant with the only jumps at the origin and infinity. So $f_k \in PC_p$ for every $p \in (1, \infty)$. It is obvious that the functions g_k belong to $\widetilde{SO}^1 \setminus C(\overline{\mathbb{R}})$ because $|g_k(t)| < 1$ for all $t \in \mathbb{R}$,

$$\lim_{t\to\infty} tg_k'(t) = \lim_{t\to\infty}\left(\frac{2}{t^2+1} + \frac{kt^{2k}}{t^{2k}+1}\cdot\frac{i}{\sqrt{\log(t^{2k}+1)}}\right)g_k(t) = 0$$

and g_k do not have limits as $t \to \pm\infty$. Since g_k are even, from Lemmas 2.1–2.2 it follows that $g_k \in SO^1 \subset SO_p$ for all $p \in (1,\infty)$. By Lemma 2.8, for every functional $\eta \in M_\infty(SO)$ there exists a sequence $\tau_n > 1$, $\tau_n \to +\infty$ such that

$$\eta(g_k) = \lim_{n\to\infty} g_k(\tau_n) \quad \text{for all} \quad k \in \mathbb{N}.$$

Notice that $\eta(g_k) \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

For every $m \in \mathbb{N}$, the function

$$a = \sum_{k=1}^{m} f_k g_k$$

belongs to the algebra $[PC_p, SO_p]$ and

$$(\alpha_\eta a)(-\infty) = \sum_{k=1}^{m} (\alpha_\eta f_k)(-\infty) \cdot (\alpha_\eta g_k)(-\infty) = \sum_{k=1}^{m} c_k \eta(g_k),$$

$$(\alpha_\eta a)(+\infty) = \sum_{k=1}^{m} (\alpha_\eta f_k)(+\infty) \cdot (\alpha_\eta g_k)(+\infty) = \sum_{k=1}^{m} d_k \eta(g_k).$$

Thus $\gamma_\eta a$ is the piecewise constant function

$$\gamma_\eta a = \sum_{k=1}^{m} (c_k \chi_- + d_k \chi_+) \eta(g_k) \in PC_p. \qquad \square$$

Let $\mathcal{C}^\pi$ denote the smallest closed subalgebra of the Calkin algebra $\mathcal{B}/\mathcal{K}$ that contains all the cosets $aI + \mathcal{K}$ with $a \in C(\dot{\mathbb{R}})$ and $W^0(b) + \mathcal{K}$ with $b \in SO_p$.

Lemma 2.9. *The algebra $\mathcal{C}^\pi$ is commutative. The maximal ideal space $M(\mathcal{C}^\pi)$ of $\mathcal{C}^\pi$ is homeomorphic to the set*

$$\Omega := \big(\mathbb{R} \times M_\infty(SO)\big) \cup \big(\{\infty\} \times \mathbb{R}\big) \cup \big(\{\infty\} \times M_\infty(SO)\big). \tag{2.8}$$

Proof. Lemma 2.7 implies that the algebra $\mathcal{C}^\pi$ is commutative. Then in the same way as in [1, Lemma 5.1] (see also [21, Proposition 14.1]) one can prove that its maximal ideal space $M(\mathcal{C}^\pi)$ is homeomorphic to Ω. $\square$

3. Singular integral operators on connected subsets of $\mathbb{R}$

3.1. Circular arcs

In this subsection we will follow [8, Section 9.1] and [3, Section 7.4]. Given two points $z_1, z_2 \in \mathbb{C}$ and a number $s \in (1,\infty)$ one can define the circular arc $\mathfrak{A}_s(z_1, z_2)$ between z_1 and z_2 by

$$\mathfrak{A}_s(z_1,z_2) := \left\{ z \in \mathbb{C} \setminus \{0,1\} : \arg \frac{z - z_1}{z - z_2} \in \frac{2\pi}{s} + 2\pi\mathbb{Z} \right\} \cup \{z_1, z_2\}.$$

If $z_1 = z_2 =: z$, then $\mathfrak{A}_s(z_1,z_2)$ is simply $\{z\}$. The set $\mathfrak{A}_2(z_1,z_2)$ is the segment $[z_1,z_2]$, if $s > 2$ (resp. $1 < s < 2$), then $\mathfrak{A}_s(z_1,z_2)$ is the circular arc at the points of which the segment $[z_1,z_2]$ is seen at the angle $2\pi/s$ (resp. $2\pi - 2\pi/s$) and which

lies on the right (resp. left) of the straight line passing first z_1 and then z_2. The arc $\mathfrak{A}_s(z_1, z_2)$ can be analytically represented by

$$z = z_1[1 - f_s(\mu)] + z_2 f_s(\mu) \quad (\mu \in [0, 1], \quad z \in \mathfrak{A}_s(z_1, z_2)),$$

where $f_s : [0, 1] \to \mathbb{C}$ is defined by

$$f_s(\mu) := \begin{cases} \mu & \text{if} \quad s = 2, \\ \dfrac{\sin(\pi\mu - 2\pi\mu/s)}{\sin(\pi - 2\pi/s)} e^{i(\pi - 2\pi/s)(\mu - 1)} & \text{if} \quad s \in (1, 2) \cup (2, \infty). \end{cases}$$

3.2. The Gohberg-Krupnik theorem

Let $-\infty \le \alpha < \beta \le +\infty$ and $(\alpha, \beta) \ne \mathbb{R}$. For $\varphi \in L^1(\alpha, \beta)$, consider the Cauchy singular integral operator $S_{(\alpha,\beta)}$ given by

$$(S_{(\alpha,\beta)}\varphi)(t) := \frac{1}{\pi i} \int_\alpha^\beta \frac{\varphi(\tau)}{\tau - t} \, d\tau,$$

where the integral is understood in the principal value sense. Put

$$P_{(\alpha,\beta)} := (I + S_{(\alpha,\beta)})/2, \quad Q_{(\alpha,\beta)} := (I - S_{(\alpha,\beta)})/2.$$

It is well known that these operators are bounded on $L^p(\alpha, \beta)$ for $1 < p < \infty$ (see e.g. [8, Chap. 1, Theorem 3.1]). By $PC[\alpha, \beta]$ denote the collection of all elements of PC with a finite number of jumps restricted to $[\alpha, \beta]$.

For $a \in PC[\alpha, \beta]$, $p \in (1, \infty)$, and $\mu \in [0, 1]$, put

$$a_p(t, \mu) := \begin{cases} [1 - f_p(\mu)] + a(\alpha^+) f_p(\mu) & \text{if} \quad t = \alpha, \\ a(t^-)[1 - f_p(\mu)] + a(t^+) f_p(\mu) & \text{if} \quad t \in (\alpha, \beta), \\ a(\beta^-)[1 - f_p(\mu)] + f_p(\mu) & \text{if} \quad t = \beta. \end{cases}$$

The range of this function is a closed continuous curve obtained from the range of the function a by adding the arcs $\mathfrak{A}_p(a(t_k^-), a(t_k^+))$ for all jumps t_k of a and the arcs $\mathfrak{A}_p(a(\beta^-), 1)$ and $\mathfrak{A}_p(1, a(\alpha^+))$. It can be oriented in the natural manner: on the intervals of continuity of a, the motion along this curve agrees with the increment of t, while the supplementary arcs $\mathfrak{A}_p(\cdot, \cdot)$ are oriented from the point on the first position to the point on the second position in the definition of the arc $\mathfrak{A}_p(\cdot, \cdot)$. If $a_p(t, \mu) \ne 0$ for all $(t, \mu) \in [\alpha, \beta] \times [0, 1]$, then by $\operatorname{wind} a_p$ we denote the winding number of this curve about the origin.

The Gohberg-Krupnik one-sided invertibility criteria for singular integral operators with piecewise continuous coefficients over the segment $[\alpha, \beta]$ read as follows (see [8, Chap. 9, Theorem 4.1] and also [14, Chap. IV, Theorems 5.1 and 6.1]).

Theorem 3.1 (Gohberg-Krupnik). *Let $-\infty \le \alpha < \beta \le +\infty$ and $(\alpha, \beta) \ne \mathbb{R}$. Suppose $1 < p < \infty$ and $c, d \in PC[\alpha, \beta]$. The operator $A = P_{(\alpha,\beta)} cI + Q_{(\alpha,\beta)} dI$ is*

at least one-sided invertible on $L^p(\alpha,\beta)$ *if and only if the following conditions are satisfied for all* $\mu\in[0,1]$*:*

$$\begin{cases} c(\alpha^+)f_p(\mu)+d(\alpha^+)[1-f_p(\mu)]\neq 0, \\ c(t^+)d(t^-)f_p(\mu)+c(t^-)d(t^+)[1-f_p(\mu)]\neq 0 \quad (t\in(\alpha,\beta)), \\ c(\beta^-)[1-f_p(\mu)]+d(\beta^-)f_p(\mu)\neq 0. \end{cases} \tag{3.1}$$

If these conditions are fulfilled, then the operator A *is invertible, invertible only from the left, invertible only from the right depending on whether the number* $\operatorname{wind}_p(c/d)$ *is equal to zero, positive, or negative, respectively.*

3.3. Spectra of the operators $P_{(\alpha,\beta)}$ and $Q_{(\alpha,\beta)}$

For $p\in(1,\infty)$, put $q:=p/(p-1)$ and define the lentiform domain $\mathfrak{L}_p$ by

$$\mathfrak{L}_p:=\big\{z\in\mathfrak{A}_s(0,1)\ :\ \min\{p,q\}\le s\le\max\{p,q\}\big\}. \tag{3.2}$$

By $\mathcal{B}(L^p(\alpha,\beta))$ denote the Banach algebra of all bounded linear operators on $L^p(\alpha,\beta)$. The spectrum of an element a in a unital Banach algebra B will be denoted by $\operatorname{sp}_B(a)$.

Theorem 3.2. *Let* $1<p<\infty$ *and* $\mathfrak{L}_p$ *be the lentiform domain given by* (3.2)*. If* $-\infty\le\alpha<\beta\le+\infty$ *and* $(\alpha,\beta)\neq\mathbb{R}$*, then*

$$\operatorname{sp}_{\mathcal{B}(L^p(\alpha,\beta))}(P_{(\alpha,\beta)})=\operatorname{sp}_{\mathcal{B}(L^p(\alpha,\beta))}(Q_{(\alpha,\beta)})=\mathfrak{L}_p.$$

Proof. Let $\lambda\in\mathbb{C}$. Then for the operator

$$A_\lambda:=P_{(\alpha,\beta)}-\lambda I=P_{(\alpha,\beta)}(1-\lambda)I+Q_{(\alpha,\beta)}(-\lambda)I$$

conditions (3.1) with $c:=1-\lambda$ and $d:=-\lambda$ have the form, for $\mu\in[0,1]$,

$$c(\alpha^+)f_p(\mu)+d(\alpha^+)[1-f_p(\mu)]=(1-\lambda)f_p(\mu)-\lambda[1-f_p(\mu)]=f_p(\mu)-\lambda\neq 0;$$

$$\begin{aligned} &c(t^+)d(t^-)f_p(\mu)+c(t^-)d(t^+)[1-f_p(\mu)] \\ &=(1-\lambda)(-\lambda)f_p(\mu)+(1-\lambda)(-\lambda)[1-f_p(\mu)]=\lambda(\lambda-1)\neq 0 \end{aligned}$$

whenever $t\in(\alpha,\beta)$; and

$$c(\beta^-)[1-f_p(\mu)]+d(\beta^-)f_p(\mu)=(1-\lambda)[1-f_p(\mu)]+(-\lambda)f_p(\mu)=1-\lambda-f_p(\mu)\neq 0.$$

These conditions are equivalent to

$$\lambda\notin\mathfrak{A}_p(0,1),\quad 1-\lambda\notin\mathfrak{A}_p(0,1). \tag{3.3}$$

By Theorem 3.1, the operator A_λ is one-sided invertible if and only if (3.3) is fulfilled. Further, if (3.3) holds, then

$$\left(\frac{c}{d}\right)_p(t,\mu)=\begin{cases} \dfrac{\lambda-1}{\lambda}f_p(\mu)+[1-f_p(\mu)] & \text{if} \quad t=\alpha, \\ \dfrac{\lambda-1}{\lambda} & \text{if} \quad t\in(\alpha,\beta), \\ \dfrac{\lambda-1}{\lambda}[1-f_p(\mu)]+f_p(\mu) & \text{if} \quad t=\beta. \end{cases}$$

So, the range of $(c/d)_p$ coincides with the closed curve $\mathfrak{A}_p\left(1,\frac{\lambda-1}{\lambda}\right)\cup\mathfrak{A}_p\left(\frac{\lambda-1}{\lambda},1\right)$. It is easy to see that

$$\operatorname{wind}\mathfrak{A}_p\left(1,\frac{\lambda-1}{\lambda}\right)\cup\mathfrak{A}_p\left(\frac{\lambda-1}{\lambda},1\right)=0$$

if and only if the origin lies outside the convex lentiform domain $\mathfrak{M}_p(\lambda)$ bounded by the arcs $\mathfrak{A}_p\left(1,\frac{\lambda-1}{\lambda}\right)=\mathfrak{A}_q\left(\frac{\lambda-1}{\lambda},1\right)$, where $1/p+1/q=1$, and $\mathfrak{A}_p\left(\frac{\lambda-1}{\lambda},1\right)$.

Let $I_p:=[\min\{p,q\},\max\{p,q\}]$. By Theorem 3.1, the operator A_λ is invertible on $L^p(\alpha,\beta)$ if and only if (3.3) is fulfilled and $0\notin\mathfrak{M}_p(\lambda)$. Then the spectrum of $P_{(\alpha,\beta)}$ is equal to $\sigma_1\cup\sigma_2\cup\sigma_2$, where

$$\sigma_1:=\big\{\lambda\in\mathbb{C}:\lambda\in\mathfrak{A}_p(0,1)\big\}=\mathfrak{A}_p(0,1),$$
$$\sigma_2:=\big\{\lambda\in\mathbb{C}:1-\lambda\in\mathfrak{A}_p(0,1)\big\}=\mathfrak{A}_p(0,1),$$

and

$$\begin{aligned}\sigma_3&:=\big\{\lambda\in\mathbb{C}\setminus(\sigma_1\cup\sigma_2)\ :\ 0\in\mathfrak{M}_p(\lambda)\big\}\\&=\left\{\lambda\in\mathbb{C}\setminus\mathfrak{A}_p(0,1)\ :0\in\bigcup_{s\in I_p}\mathfrak{A}_s\left(\frac{\lambda-1}{\lambda},1\right)\right\}\\&=\left\{\lambda\in\mathbb{C}\setminus\mathfrak{A}_p(0,1)\ :\ \frac{\lambda-1}{\lambda}[1-f_s(\mu)]+f_s(\mu)=0\text{ for some }s\in I_p\right\}\\&=\big\{\lambda\in\mathbb{C}\setminus\mathfrak{A}_p(0,1)\ :\ 1-\lambda\in\mathfrak{A}_s(0,1)\text{ for some }s\in I_p\big\}\\&=\big\{\lambda\in\mathbb{C}\setminus\mathfrak{A}_p(0,1)\ :\ 1-\lambda\in\mathfrak{L}_p(0,1)\big\}\\&=\big\{\lambda\in\mathbb{C}\setminus\mathfrak{A}_p(0,1)\ :\ \lambda\in\mathfrak{L}_p(0,1)\big\}\\&=\mathfrak{L}_p\setminus\mathfrak{A}_p(0,1).\end{aligned}$$

Thus

$$\operatorname{sp}_{\mathcal{B}(L^p(\alpha,\beta))}(P_{(\alpha,\beta)})=\sigma_1\cup\sigma_2\cup\sigma_3=\mathfrak{A}_p(0,1)\cup\mathfrak{A}_p(0,1)\cup(\mathfrak{L}_p\setminus\mathfrak{A}_p(0,1))=\mathfrak{L}_p.$$

The proof of the equality $\operatorname{sp}_{\mathcal{B}(L^p(\alpha,\beta))}(Q_{(\alpha,\beta)})=\mathfrak{L}_p$ is analogous. □

Let χ_E denote the characteristic function of a set $E\subset\mathbb{R}$.

Corollary 3.3. *Let $1<p<\infty$ and $\mathfrak{L}_p$ be the lentiform domain given by* (3.2). *Then*

(a) $\operatorname{sp}_{\mathcal{B}}\left(\chi_{(-1,0)}P_{\mathbb{R}}\chi_{(-1,0)}I+\chi_{(-\infty,-1)}Q_{\mathbb{R}}\chi_{(-\infty,-1)}I\right)=\mathfrak{L}_p$;

(b) $\operatorname{sp}_{\mathcal{B}}\left(\chi_{(1,\infty)}P_{\mathbb{R}}\chi_{(1,\infty)}I+\chi_{(0,1)}Q_{\mathbb{R}}\chi_{(0,1)}I\right)=\mathfrak{L}_p$;

(c) $\operatorname{sp}_{\mathcal{B}}\left(\chi_+P_{\mathbb{R}}\chi_+I+\chi_-Q_{\mathbb{R}}\chi_-I\right)=\mathfrak{L}_p$.

Proof. Let us prove equality (a). The operator

$$\begin{aligned}&\chi_{(-1,0)}P_{\mathbb{R}}\chi_{(-1,0)}I+\chi_{(-\infty,-1)}Q_{\mathbb{R}}\chi_{(-\infty,-1)}I-\lambda I\\&=\chi_{(-\infty,-1)}(Q_{\mathbb{R}}-\lambda I)\chi_{(-\infty,-1)}I+\chi_{(-1,0)}(P_{\mathbb{R}}-\lambda I)\chi_{(-1,0)}I-\lambda\chi_+I\end{aligned}$$

is represented in the direct sum

$$L^p(-\infty,-1) \dot{+} L^p(-1,0) \dot{+} L^p(\mathbb{R}_+) = L^p(\mathbb{R})$$

by the matrix

$$\begin{bmatrix} Q_{(-\infty,-1)} - \lambda I & 0 & 0 \\ 0 & P_{(-1,0)} - \lambda I & 0 \\ 0 & 0 & -\lambda I \end{bmatrix}.$$

Hence, by Theorem 3.2,

$$\begin{aligned} &\mathrm{sp}_{\mathcal{B}}\left(\chi_{(-1,0)} P_{\mathbb{R}} \chi_{(-1,0)} I + \chi_{(-\infty,-1)} Q_{\mathbb{R}} \chi_{(-\infty,-1)} I\right) \\ &= \mathrm{sp}_{\mathcal{B}(L^p(-\infty,-1))}(Q_{(-\infty,-1)}) \cup \mathrm{sp}_{\mathcal{B}(L^p(-1,0))}(P_{(-1,0)}) \cup \mathrm{sp}_{\mathcal{B}(L^p(\mathbb{R}_+))}(0) \\ &= \mathfrak{L}_p \cup \mathfrak{L}_p \cup \{0\} = \mathfrak{L}_p. \end{aligned}$$

The proof of equalities (b)–(c) is analogous. □

3.4. Singular integral operator with piecewise constant coefficients

Let z_1, z_2, z_3 be an ordered triple of points in the complex plane. It is clear that

$$\mathfrak{A}_p(z_1,z_2) \cup \mathfrak{A}_p(z_2,z_3) \cup \mathfrak{A}_p(z_3,z_1) =: \mathfrak{C}_p(z_1,z_2,z_3)$$

is a closed curve in the complex plane (which degenerates to the point z if $z = z_1 = z_2 = z_3$). Each arc $\mathfrak{A}_p(z_1,z_2)$, $\mathfrak{A}_p(z_2,z_3)$, and $\mathfrak{A}_p(z_3,z_1)$ can be naturally oriented starting from the point on the first position and terminating at the point on the second position in the definition of the arc $\mathfrak{A}_p(\cdot,\cdot)$. This orientation induces the orientation of the curve $\mathfrak{C}_p(z_1,z_2,z_3)$. If $0 \notin \mathfrak{C}_p(z_1,z_2,z_3)$, then by $\mathrm{wind}\, \mathfrak{C}_p(z_1,z_2,z_3)$ we denote the winding number of the curve $\mathfrak{C}_p(z_1,z_2,z_3)$ about the origin. By definition, $\mathrm{wind}\, \mathfrak{C}_p(z,z,z) = 0$.

Lemma 3.4. *Let $1 < p < \infty$. Suppose $a_-, a_+, b_-, b_+ \in \mathbb{C}$. The singular integral operator*

$$P_{(-1,1)}(a_-\chi_{(-1,0)} + b_-\chi_{(0,1)})I + Q_{(-1,1)}(a_+\chi_{(-1,0)} + b_+\chi_{(0,1)})I$$

is invertible on the space $L^p(-1,1)$ if and only if

$$0 \notin \{a_-, a_+, b_-, b_+\}, \quad \mathrm{wind}\, \mathfrak{C}_p\left(1, \frac{a_-}{a_+}, \frac{b_-}{b_+}\right) = 0. \tag{3.4}$$

This lemma follows from Theorem 3.1. Its proof is similar to the proof of Theorem 3.2 and it is omitted here.

4. Algebraization

4.1. Stability of the finite section method

Let $\mathcal{E}$ be the set formed by all the sequences (A_τ) (depending on a parameter $\tau \in \mathbb{R}_+$) of operators $A_\tau \in \mathcal{B}$ such that

$$\sup_{\tau \in \mathbb{R}_+} \|A_\tau\|_{\mathcal{B}} < \infty.$$

One says that the sequence $(A_\tau) \in \mathcal{E}$ is *stable* if there exists a positive constant τ_0 such that the operator $A_\tau \in \mathcal{B}$ is invertible for any $\tau > \tau_0$ and

$$\sup_{\tau > \tau_0} \|A_\tau^{-1}\|_{\mathcal{B}} < \infty.$$

The following result is well known (see e.g. [10, Proposition 1.1] and also [5, Proposition 7.3]).

Theorem 4.1 (Polski). *The finite section method applies to an operator $A \in \mathcal{B}$ if and only if A is invertible and the sequence $(P_\tau A P_\tau + Q_\tau)$ is stable.*

4.2. Algebra $\mathcal{E}$ and its ideal $\mathcal{G}$

Lemma 4.2. *The set $\mathcal{E}$ with the operations*

$$(A_\tau) + (B_\tau) := (A_\tau + B_\tau), \quad (A_\tau)(B_\tau) := (A_\tau B_\tau), \quad \lambda(A_\tau) := (\lambda A_\tau) \quad (\lambda \in \mathbb{C}),$$

the identity element (I), and the norm

$$\|(A_\tau)\|_{\mathcal{E}} := \sup_{\tau \in \mathbb{R}_+} \|A_\tau\|_{\mathcal{B}}$$

forms a unital Banach algebra.

This statement is proved as [11, Proposition 1.13].

Note that the constant sequences (A) are included in $\mathcal{E}$ for any $A \in \mathcal{B}$.

Definition 4.3. By $\mathcal{A}$ denote the smallest Banach subalgebra of $\mathcal{E}$ that contains all constant sequences (aI) with $a \in PC$ and $(W^0(b))$ with $b \in [PC_p, SO_p]$ and the sequence (P_τ).

We will be interested in conditions for the stability of any sequence in $\mathcal{A}$. Clearly, $\mathcal{A}$ contains the sequences (1.3).

Let $\mathcal{G}$ be the set of all sequences $(A_\tau) \in \mathcal{E}$ satisfying

$$\lim_{\tau \to \infty} \|A_\tau\|_{\mathcal{B}} = 0.$$

Lemma 4.4. *The set $\mathcal{G}$ is a closed two-sided ideal of the algebra $\mathcal{E}$.*

The proof is similar to the proof of [11, Proposition 1.14].

Theorem 4.5 (Kozak). *Let $(A_\tau) \in \mathcal{E}$. The sequence (A_τ) is stable if and only if the coset $(A_\tau) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{E}/\mathcal{G}$.*

This result is well known (see e.g. [5, Proposition 7.3], [10, Proposition 1.2], [11, Theorem 1.15]).

5. Essentialization

5.1. Algebra $\mathcal{F}$ and its ideal $\mathcal{G}$

A sequence of operators on a Banach space is said to converge **-strongly* if it converges strongly and the sequence of the adjoint operators converges strongly on the dual space.

Let $\tau \in \mathbb{R}_+$. By V_τ denote the additive shift operator given by

$$(V_\tau f)(x) := f(x-\tau) \quad (x \in \mathbb{R}).$$

It is clear that V_τ is bounded and invertible on $L^p(\mathbb{R})$ and $V_\tau^{-1} = V_{-\tau}$. Moreover, $\|V_\tau\|_{\mathcal{B}} = 1$ for all $\tau \in \mathbb{R}$.

Let $\mathcal{F}$ denote the set of all sequences $\mathbf{A} := (A_\tau) \in \mathcal{E}$ such that the sequences (A_τ), $(V_{-\tau}A_\tau V_\tau)$, and $(V_\tau A_\tau V_{-\tau})$ are *-strongly convergent as $\tau \to +\infty$.

Lemma 5.1. (a) *The set $\mathcal{F}$ is a closed unital subalgebra of the algebra $\mathcal{E}$.*

(b) *Let $i \in \{-1, 0, 1\}$. The mappings $\mathsf{W}_i : \mathcal{F} \to \mathcal{B}$ given by*

$$\begin{aligned} \mathsf{W}_{-1}(\mathbf{A}) &:= \operatorname*{s-lim}_{\tau\to+\infty} V_\tau A_\tau V_{-\tau}, \\ \mathsf{W}_0(\mathbf{A}) &:= \operatorname*{s-lim}_{\tau\to+\infty} A_\tau, \\ \mathsf{W}_1(\mathbf{A}) &:= \operatorname*{s-lim}_{\tau\to+\infty} V_{-\tau} A_\tau V_\tau \end{aligned}$$

for $\mathbf{A} = (A_\tau) \in \mathcal{F}$ are bounded unital homomorphisms with the norms

$$\|\mathsf{W}_{-1}\| = \|\mathsf{W}_0\| = \|\mathsf{W}_1\| = 1.$$

(c) *The set $\mathcal{G}$ is a closed two-sided ideal of the algebra $\mathcal{F}$.*

(d) *The ideal $\mathcal{G}$ lies in the kernel of each homomorphism W_i for $i \in \{-1, 0, 1\}$.*

(e) *The algebra $\mathcal{F}$ is inverse closed in the algebra $\mathcal{E}$ and the algebra $\mathcal{F}/\mathcal{G}$ is inverse closed in the algebra $\mathcal{E}/\mathcal{G}$.*

The proof follows the proof of [20, Proposition 4.1]. Notice that the algebra $\mathcal{F}$ in the present paper is larger than the algebra considered in [20] and also denoted there by $\mathcal{F}$ (see also Remark 7.3).

5.2. The algebra $\mathcal{A}$ is contained in the algebra $\mathcal{F}$

Proposition 5.2. (a) *If $\mathbf{P} = (P_\tau)$, then $\mathbf{P} \in \mathcal{F}$ and*

$$\mathsf{W}_{-1}(\mathbf{P}) = \chi_+ I, \quad \mathsf{W}_0(\mathbf{P}) = I, \quad \mathsf{W}_1(\mathbf{P}) = \chi_- I. \tag{5.1}$$

(b) *If $\mathbf{A} = (aI)$ with $a \in PC$, then $\mathbf{A} \in \mathcal{F}$ and*

$$\mathsf{W}_{-1}(\mathbf{A}) = a(-\infty)I, \quad \mathsf{W}_0(\mathbf{A}) = aI, \quad \mathsf{W}_1(\mathbf{A}) = a(+\infty)I. \tag{5.2}$$

(c) *If $\mathbf{B} = (W^0(b))$ with $b \in [PC_p, SO_p]$, then $\mathbf{B} \in \mathcal{F}$ and*

$$\mathsf{W}_{-1}(\mathbf{B}) = W^0(b), \quad \mathsf{W}_0(\mathbf{B}) = W^0(b), \quad \mathsf{W}_1(\mathbf{B}) = W^0(b). \tag{5.3}$$

Proof. The proof of equalities (5.1) and (5.2) is straightforward, equalities in (5.3) are trivial because the convolution operator $W^0(b)$ is translation-invariant, that is, $V_{\pm\tau}W^0(b)V_{\mp\tau} = W^0(b)$ for every $b \in \mathcal{M}_p$, in particular, for every $b \in [PC_p, SO_p]$. To finish the proof, it remains to note that

$$P_\tau^* = P_\tau \in \mathcal{B}(L^q(\mathbb{R})), \quad [aI]^* = \overline{a}I \in \mathcal{B}(L^q(\mathbb{R})), \quad [W^0(b)]^* = W^0(\overline{b}) \in \mathcal{B}(L^q(\mathbb{R})),$$

where $1/p + 1/q = 1$. Therefore the existence of the strong limits for the adjoint operators in the definition of the algebra $\mathcal{F}$ can be obtained in the same way as the existence of the strong limits in (5.1)–(5.3). □

Corollary 5.3. *The algebra $\mathcal{A}$ is a closed unital subalgebra of the algebra $\mathcal{F}$.*

5.3. Ideals $\mathcal{J}_{-1}$, $\mathcal{J}_0$, $\mathcal{J}_1$, and $\mathcal{J}$ of the algebra $\mathcal{F}$

Recall that $\mathcal{K}$ denotes the ideal of the compact operators on $L^p(\mathbb{R})$. Put

$$\begin{aligned} \mathcal{J}_{-1} &:= \big\{(V_{-\tau}KV_\tau) + (G_\tau) \ : \ K \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\big\}, \\ \mathcal{J}_0 &:= \big\{(K) + (G_\tau) \ : \ K \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\big\}, \\ \mathcal{J}_1 &:= \big\{(V_\tau KV_{-\tau}) + (G_\tau) \ : \ K \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\big\}, \end{aligned}$$

and

$$\mathcal{J} := \big\{(V_\tau K_1 V_{-\tau}) + (K_0) + (V_{-\tau}K_{-1}V_\tau) + (G_\tau) \ : \ K_{-1}, K_0, K_1 \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\big\}.$$

Lemma 5.4. *The sets $\mathcal{J}_i$ with $i \in \{-1, 0, 1\}$ and $\mathcal{J}$ are closed two-sided ideals of the algebra $\mathcal{F}$.*

Proof. First we note that $(V_{\pm\tau})$ converges weakly to zero as $\tau \to +\infty$. Let us show that $\mathcal{J}_{-1} \subset \mathcal{F}$. Let $\mathbf{J} = (J_\tau) = (V_{-\tau}KV_\tau) + (G_\tau)$ for some $K \in \mathcal{K}$ and $(G_\tau) \in \mathcal{G}$. Since K is compact and $V_{\pm\tau}^2 = V_{\pm 2\tau}$, we obtain that (J_τ) is *-strongly convergent to zero, $(V_{-\tau}J_\tau V_\tau)$ is *-strongly convergent to zero, and $(V_\tau J_\tau V_{-\tau})$ is *-strongly convergent to K. Thus $\mathcal{J}_{-1} \subset \mathcal{F}$.

Let $\mathbf{A} = (A_\tau) \in \mathcal{F}$. Then

$$\begin{aligned} \mathbf{AJ} &= (A_\tau V_{-\tau}KV_\tau) + (A_\tau G_\tau) \\ &= (V_{-\tau}V_\tau A_\tau V_{-\tau}KV_\tau) + (A_\tau G_\tau) \\ &= (V_{-\tau}\mathsf{W}_{-1}(\mathbf{A})KV_\tau) + (V_{-\tau}[V_\tau A_\tau V_{-\tau}K - \mathsf{W}_{-1}(\mathbf{A})K]V_\tau + A_\tau G_\tau) \\ &=: (V_{-\tau}\mathsf{W}_{-1}(\mathbf{A})KV_\tau) + (G_\tau'). \end{aligned}$$

Since the sequence $(V_\tau A_\tau V_{-\tau} - \mathsf{W}_{-1}(\mathbf{A}))$ converges strongly to zero and $K \in \mathcal{K}$, the sequence $(V_\tau A_\tau V_{-\tau}K - \mathsf{W}_{-1}(\mathbf{A})K)$ converges uniformly to zero. Thus

$$(G_\tau') := (V_{-\tau}[V_\tau A_\tau V_{-\tau}K - \mathsf{W}_{-1}(\mathbf{A})K]V_{-\tau} + A_\tau G_\tau) \in \mathcal{G}$$

and $\mathbf{AJ} \in \mathcal{J}_{-1}$. Analogously one can show that $\mathbf{JA} \in \mathcal{J}_{-1}$. This means that $\mathcal{J}_{-1}$ is a two-sided ideal of $\mathcal{F}$.

Let us show that $\mathcal{J}_{-1}$ is closed. Suppose $(\mathbf{J}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{J}_{-1}$. Since $\mathcal{F}$ is closed in $\mathcal{E}$ by Lemma 5.1(a), the sequence $(\mathbf{J}_k)_{k\in\mathbb{N}}$ is convergent

to some $\mathbf{J} = (J_\tau) \in \mathcal{F}$. We claim that $\mathbf{J} \in \mathcal{J}_{-1}$. Dy definition of $\mathcal{J}_{-1}$, there exist sequences $(K^{(j)})_{j\in\mathbb{N}} \subset \mathcal{K}$ and $(G_\tau^{(j)})_{j\in\mathbb{N}} \subset \mathcal{G}$ such that

$$\mathbf{J}_j = (J_\tau^{(j)}) = (V_{-\tau}K^{(j)}V_\tau) + (G_\tau^{(j)}).$$

From Lemma 5.1(b) it follows that for all $j, k \in \mathbb{N}$,

$$\|K^{(j)} - K^{(k)}\|_{\mathcal{B}} = \|\mathsf{W}_{-1}(\mathbf{J}_j) - \mathsf{W}_{-1}(\mathbf{J}_k)\|_{\mathcal{B}} \le \|\mathbf{J}_j - \mathbf{J}_k\|_{\mathcal{E}}.$$

Then $(K^{(j)})_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}$. Let $K \in \mathcal{K}$ be its limit. Put

$$\mathbf{G} := (G_\tau) := (J_\tau) - (V_{-\tau}KV_\tau).$$

Taking into account that $\|V_{\pm\tau}\|_{\mathcal{B}} = 1$, we obtain

$$\begin{aligned}\|(G_\tau) - (G_\tau^{(j)})\|_{\mathcal{E}} &\le \|(J_\tau) - (J_\tau^{(j)})\|_{\mathcal{E}} + \|(V_{-\tau}KV_\tau) - (V_{-\tau}K^{(j)}V_\tau)\|_{\mathcal{E}} \\ &\le \|\mathbf{J} - \mathbf{J}_j\|_{\mathcal{E}} + \|K - K^{(j)}\|_{\mathcal{B}}.\end{aligned}$$

Hence $\mathbf{G}$ is the limit of $(\mathbf{G}_j)_{j\in\mathbb{N}}$ with $\mathbf{G}_j := (G_\tau^{(j)}) \in \mathcal{G}$ in the norm of $\mathcal{E}$. By Lemma 5.1(c), $\mathbf{G} \in \mathcal{G}$. Thus $\mathbf{J} \in \mathcal{J}_{-1}$ and $\mathcal{J}_{-1}$ is closed.

Analogously it can be shown that $\mathcal{J}_0$, $\mathcal{J}_{-1}$, and $\mathcal{J}$ are closed two-sided ideals of the algebra $\mathcal{F}$. ☐

5.4. Lifting

The main result of this section is the following lifting theorem adapted for our purposes from [10, Theorem 1.8].

Theorem 5.5. *Let $\mathbf{A} = (A_\tau) \in \mathcal{F}$. The coset $\mathbf{A} + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$ if and only if the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$, and $\mathsf{W}_1(\mathbf{A})$ are invertible in $\mathcal{B}$ and the coset $\mathbf{A} + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{J}$.*

Proof. The proof is analogous to the proof of the abstract lifting theorem [10, Theorem 1.8], although we consider the invertibility of $\mathsf{W}_i(\mathbf{A})$ in $\mathcal{B}$, while there it is considered in the images of the homomorphisms W_i.

Necessity. If $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$, then there exist sequences $\mathbf{B} \in \mathcal{F}$ and $\mathbf{G}_1, \mathbf{G}_2 \in \mathcal{G}$ such that

$$\mathbf{AB} = \mathbf{I} + \mathbf{G}_1, \quad \mathbf{BA} = \mathbf{I} + \mathbf{G}_2. \tag{5.4}$$

Let $i \in \{-1, 0, 1\}$. From Lemma 5.1(b),(d) we know that $\mathsf{W}_i : \mathcal{F} \to \mathcal{B}$ is a bounded unital homomorphism and $\mathsf{W}_i(\mathbf{G}_1) = \mathsf{W}_i(\mathbf{G}_2) = 0$. Applying this homomorphism to the equalities in (5.4), we conclude that the operator $\mathsf{W}_i(\mathbf{A})$ is invertible in $\mathcal{B}$ and its inverse is $\mathsf{W}_i(\mathbf{B})$. From the definition of the ideal $\mathcal{J}$ it follows that $\mathbf{G}_1, \mathbf{G}_2 \in \mathcal{J}$. Thus the equalities (5.4) imply also that $\mathbf{A} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$. The necessity part is proved.

Sufficiency. Assume that the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$, $\mathsf{W}_1(\mathbf{A})$ are invertible in $\mathcal{B}$ and the coset $\mathbf{A} + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{J}$. Then there exist sequences $\mathbf{B} \in \mathcal{F}$ and $\mathbf{J} \in \mathcal{J}$ such that $\mathbf{BA} = \mathbf{I} + \mathbf{J}$. From the definition

of $\mathcal{J}$ we conclude that there exist elements $\mathbf{J}_i \in \mathcal{J}_i$ such that $\mathbf{J} = \mathbf{J}_{-1} + \mathbf{J}_0 + \mathbf{J}_1$. Taking into account that, by Lemma 5.1(d),

$$\begin{aligned} &\mathsf{W}_{-1}[(V_{-\tau}KV_\tau) + (G_\tau)] = K, \\ &\mathsf{W}_0[(K) + (G_\tau)] = K, \\ &\mathsf{W}_1[(V_\tau KV_{-\tau}) + (G_\tau)] = K \end{aligned} \tag{5.5}$$

for every $K \in \mathcal{K}$ and every $(G_\tau) \in \mathcal{G}$, we have that W_i maps the ideal $\mathbf{J}_i$ onto the the ideal $\mathcal{K}$ for $i \in \{-1, 0, 1\}$. Since $\mathsf{W}_i(\mathbf{J}_i) \in \mathcal{K}$, we obviously have $\mathsf{W}_i(\mathbf{J}_i)\mathsf{W}_i(\mathbf{A})^{-1} \in \mathcal{K}$. Consequently for every $i \in \{-1, 0, 1\}$, there exists $\mathbf{J}'_i \in \mathcal{J}_i$ such that

$$\mathsf{W}_i(\mathbf{J}'_i) = \mathsf{W}_i(\mathbf{J}_i)\mathsf{W}_i(\mathbf{A})^{-1}. \tag{5.6}$$

Put $\mathbf{B}' := \mathbf{B} - \mathbf{J}'_{-1} - \mathbf{J}'_0 - \mathbf{J}'_1$. Then $\mathbf{B}' + \mathcal{J} = \mathbf{B} + \mathcal{J}$ and

$$\begin{aligned} \mathbf{B}'\mathbf{A} &= \mathbf{I} + \mathbf{J} - \mathbf{J}'_{-1}\mathbf{A} - \mathbf{J}'_0\mathbf{A} - \mathbf{J}'_1\mathbf{A} \\ &= \mathbf{I} + (\mathbf{J}_{-1} - \mathbf{J}'_{-1}\mathbf{A}) + (\mathbf{J}_0 - \mathbf{J}'_0\mathbf{A}) + (\mathbf{J}_1 - \mathbf{J}'_1\mathbf{A}). \end{aligned} \tag{5.7}$$

From (5.6) it follows that $\mathsf{W}_i(\mathbf{J}_i) = \mathsf{W}_i(\mathbf{J}'_i)\mathsf{W}_i(\mathbf{A}) = \mathsf{W}_i(\mathbf{J}'_i\mathbf{A})$. Hence

$$\mathsf{W}_i(\mathbf{J}_i - \mathbf{J}'_i\mathbf{A}) = 0.$$

From this observation and (5.5) we conclude that $\mathbf{J}_i - \mathbf{J}'_i\mathbf{A} \in \mathcal{G}$ for $i \in \{-1, 0, 1\}$. Thus (5.7) can be written as

$$\mathbf{B}'\mathbf{A} = \mathbf{I} + \mathbf{G}$$

with $\mathbf{G} = (\mathbf{J}_{-1} - \mathbf{J}'_{-1}\mathbf{A}) + (\mathbf{J}_0 - \mathbf{J}'_0\mathbf{A}) + (\mathbf{J}_1 - \mathbf{J}'_1\mathbf{A}) \in \mathcal{G}$. This means that $\mathbf{A} + \mathcal{G}$ is left-invertible in $\mathcal{F}/\mathcal{G}$. Analogously it can be shown that $\mathbf{A} + \mathcal{G}$ is right-invertible in $\mathcal{F}/\mathcal{G}$. $\square$

Remark 5.6. A detailed discussion of various versions of abstract lifting theorems and their applications in the setting of Banach algebras and C^*-algebras is contained in [19, Chap. 6]. In particular, the above theorem follows from the Banach algebra inverse closed lifting theorem (see [19, Theorem 6.2.8]).

6. Localization

6.1. Allan-Douglas local principle

Recall that the *center* of an algebra A consists of all elements $a \in A$ such that $ab = ba$ for all $b \in A$. Let A be a Banach algebra with identity. By a *central subalgebra* C of A one means a subalgebra of the center of A.

Theorem 6.1 ([5, Theorem 1.35(a)] **or** [10, Theorem 1.5])**.** *Let A be a Banach algebra with identity e and let C be closed central subalgebra of A containing e. Let $M(C)$ be the maximal ideal space of C, and for $\omega \in M(C)$, let I_ω refer to the smallest closed two-sided ideal of A containing the ideal ω. Then an element $a \in A$ is invertible in A if and only if $a + I_\omega$ is invertible in the quotient algebra A/I_ω for all $\omega \in M(C)$.*

6.2. Sequences of local type

We say that a sequence $(A_\tau) \in \mathcal{F}$ is of *local type* if

$$(A_\tau)(fI) - (fI)(A_\tau) \in \mathcal{J}, \quad (A_\tau)(W^0(g)) - (W^0(g))(A_\tau) \in \mathcal{J}$$

for all $f \in C(\dot{\mathbb{R}})$ and all $g \in SO_p$. Let $\mathcal{L}$ denote the set of all sequences of local type.

Lemma 6.2. (a) *The set $\mathcal{L}$ is a closed unital subalgebra of the algebra $\mathcal{F}$.*
(b) *The set $\mathcal{J}$ is a closed two-sided ideal of the algebra $\mathcal{L}$.*
(c) *The algebra $\mathcal{L}/\mathcal{J}$ is inverse closed in the algebra $\mathcal{F}/\mathcal{J}$.*
(d) *The algebra $\mathcal{L}$ is inverse closed in the algebra $\mathcal{F}$.*

Proof. (a) By definition, $\mathcal{L} \subset \mathcal{F}$. It is clear that the sequence (I) is the identity of $\mathcal{L}$. Hence $\mathcal{L}$ is a unital subalgebra of $\mathcal{F}$. Let us show that $\mathcal{L}$ is closed. Suppose $(\mathbf{A}_j)_{j\in\mathbb{N}}$ with $\mathbf{A}_j = (A_\tau^{(j)}) \in \mathcal{L}$ is a Cauchy sequence in $\mathcal{L}$. Let $\mathbf{A} = (A_\tau) \in \mathcal{F}$ be its limit. We show that $\mathbf{A} \in \mathcal{L}$. By definition of $\mathcal{L}$,

$$\mathbf{A}_j(fI) - (fI)\mathbf{A}_j \in \mathcal{J}, \quad \mathbf{A}_j(W^0(g)) - (W^0(g))\mathbf{A}_j \in \mathcal{J} \tag{6.1}$$

for all $j \in \mathbb{N}$, $f \in C(\dot{\mathbb{R}})$, and $g \in SO_p$. Passing to the limit in (6.1) as $j \to \infty$ and taking into account that $\mathcal{J}$ is closed in $\mathcal{F}$, we conclude that $\mathbf{A}(fI) - (fI)\mathbf{A} \in \mathcal{J}$ and $\mathbf{A}(W^0(g)) - (W^0(g))\mathbf{A} \in \mathcal{J}$ for all $f \in C(\dot{\mathbb{R}})$ and $g \in SO_p$. Thus $\mathbf{A} \in \mathcal{L}$. Part (a) is proved.

(b) Since $\mathcal{J} \subset \mathcal{L} \subset \mathcal{F}$ and $\mathcal{J}$ is a closed two-sided ideal of the algebra $\mathcal{F}$, we also observe that $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{L}$. Part (b) is proved.

(c) Let $\mathbf{A} = (A_\tau) \in \mathcal{L}$ and $\mathbf{A} + \mathcal{J}$ be invertible in $\mathcal{F}/\mathcal{J}$. Then there exist sequences $\mathbf{B} = (B_\tau) \in \mathcal{F}$ and $\mathbf{J}_1 = (J_\tau^{(1)})$, $\mathbf{J}_2 = (J_\tau^{(2)})$ belonging to $\mathcal{J}$ and such that

$$I - A_\tau B_\tau = J_\tau^{(1)}, \quad I - B_\tau A_\tau = J_\tau^{(2)}.$$

Let C be one of the operators fI with $f \in C(\dot{\mathbb{R}})$ or $W^0(g)$ with $g \in SO_p$. Then

$$C = CA_\tau B_\tau + CJ_\tau^{(1)} = A_\tau CB_\tau + [CA_\tau - A_\tau C]B_\tau + CJ_\tau^{(1)}$$

and

$$\begin{aligned} B_\tau C - CB_\tau &= B_\tau A_\tau CB_\tau + B_\tau[CA_\tau - A_\tau C]B_\tau + B_\tau CJ_\tau^{(1)} - CB_\tau \\ &= [CB_\tau - J_\tau^{(2)}CB_\tau] + B_\tau[CA_\tau - A_\tau C]B_\tau + B_\tau CJ_\tau^{(1)} - CB_\tau \\ &= B_\tau[CA_\tau - A_\tau C]B_\tau + [B_\tau CJ_\tau^{(1)} - J_\tau^{(2)}CB_\tau]. \end{aligned}$$

If we put $\mathbf{C} = (C)$, then $\mathbf{BC} - \mathbf{CB} = \mathbf{B}[\mathbf{CA} - \mathbf{AC}]\mathbf{B} + \mathbf{J}$, where $\mathbf{CA} - \mathbf{AC} \in \mathcal{J}$ and $\mathbf{J} := \mathbf{BCJ}_1 - \mathbf{J}_2\mathbf{CB} \in \mathcal{J}$. Thus $\mathbf{BC} - \mathbf{CB} \in \mathcal{J}$ and $\mathbf{B} + \mathcal{J} \in \mathcal{L}/\mathcal{J}$. Part (c) is proved.

(d) Part (d) follows from part (c). □

Remark 6.3. The algebra $\mathcal{L}$ is larger than the algebra of sequences of local type considered in [20] and denoted there by $\mathcal{F}_0$ (see also Remark 7.3).

6.3. Sequences in $\mathcal{A}$ are of local type

Theorem 6.4. *The algebra $\mathcal{A}$ is a closed unital subalgebra of $\mathcal{L}$.*

Proof. The idea of the proof is borrowed from [20, Proposition 4.11].

From Corollary 5.3 we know that $\mathcal{A} \subset \mathcal{F}$. Suppose $f \in C(\dot{\mathbb{R}})$ and $g \in SO_p$. It is sufficient to prove that (aI) with $a \in PC$, $(W^0(b))$ with $b \in [PC_p, SO_p]$, and (P_τ) commute with (fI) and $(W^0(g))$ modulo the ideal $\mathcal{J}$.

Obviously aI commutes with fI. By Lemma 2.7, $aW^0(g) - W^0(g)aI$ is compact. Therefore $(aI)(fI) - (fI)(aI) \in \mathcal{J}$ and $(aI)(W^0(g)) - (W^0(g))(aI) \in \mathcal{J}$. Hence $(aI) \in \mathcal{L}$.

It is clear that $W^0(b)$ commutes with $W^0(g)$. Applying Lemma 2.7 once again, we see that $fW^0(b) - W^0(b)fI$ is compact. Then $(W^0(b))(fI) - (fI)(W^0(b)) \in \mathcal{J}$ and $(W^0(b))(W^0(g)) - (W^0(g))(W^0(b)) \in \mathcal{J}$. Thus $(W^0(b)) \in \mathcal{L}$.

Since P_τ commutes with fI, it is trivial that $(P_\tau)(fI) - (fI)(P_\tau) \in \mathcal{J}$. Consider

$$\begin{aligned}
(P_\tau W^0(g) - W^0(g)P_\tau) &= (P_\tau W^0(g)Q_\tau - Q_\tau W^0(g)P_\tau) \\
&= (P_\tau\chi_+W^0(g)\chi_+Q_\tau - Q_\tau\chi_+W^0(g)\chi_+P_\tau) \\
&\quad +(P_\tau\chi_+W^0(g)\chi_-Q_\tau - Q_\tau\chi_+W^0(g)\chi_-P_\tau) \\
&\quad +(P_\tau\chi_-W^0(g)\chi_+Q_\tau - Q_\tau\chi_-W^0(g)\chi_+P_\tau) \\
&\quad +(P_\tau\chi_-W^0(g)\chi_-Q_\tau - Q_\tau\chi_-W^0(g)\chi_-P_\tau). \qquad (6.2)
\end{aligned}$$

By Lemma 2.5, the operators $\chi_+W^0(g)\chi_-I$ and $\chi_-W^0(g)\chi_+I$ are compact. Since (Q_τ) converges *-strongly to zero, we conclude that the sequences of the second and the third line on the right-hand side of (6.2) belong to $\mathcal{G}$ and thus to $\mathcal{J}$. It remains to show that

$$(P_\tau\chi_+W^0(g)\chi_+Q_\tau - Q_\tau\chi_+W^0(g)\chi_+P_\tau) \in \mathcal{J}, \qquad (6.3)$$
$$(P_\tau\chi_-W^0(g)\chi_-Q_\tau - Q_\tau\chi_-W^0(g)\chi_-P_\tau) \in \mathcal{J}. \qquad (6.4)$$

It is easy to see that

$$\begin{aligned}
V_{-\tau}P_\tau\chi_+I &= \chi_{(-\tau,0)}V_{-\tau}, \quad & \chi_+Q_\tau V_\tau &= V_\tau\chi_+I, \\
V_{-\tau}Q_\tau\chi_+I &= \chi_+V_{-\tau}, & \chi_+P_\tau V_\tau &= V_\tau\chi_{(-\tau,0)}I,
\end{aligned}$$

where $\chi_{(-\tau,0)}$ denotes the characteristic function of the interval $(-\tau, 0)$, and

$$V_\tau V_{-\tau} = I, \quad V_{-\tau}W^0(g)V_\tau = W^0(g).$$

Then

$$\begin{aligned}
&(P_\tau\chi_+W^0(g)\chi_+Q_\tau - Q_\tau\chi_+W^0(g)\chi_+P_\tau) \\
&= (V_\tau V_{-\tau}[P_\tau\chi_+W^0(g)\chi_+Q_\tau - Q_\tau\chi_+W^0(g)\chi_+P_\tau]V_\tau V_{-\tau}) \\
&= (V_\tau[\chi_{(-\tau,0)}V_{-\tau}W^0(g)V_\tau\chi_+I - \chi_+V_{-\tau}W^0(g)V_\tau\chi_{(-\tau,0)}I]V_{-\tau})
\end{aligned}$$

$$\begin{aligned}
&= (V_\tau[\chi_- W^0(g)\chi_+ I - \chi_+ W^0(g)\chi_- I]V_{-\tau}) \\
&\quad +(V_\tau[\chi_{(-\tau,0)}I - \chi_- I]\chi_- W^0(g)\chi_+ V_{-\tau}) \\
&\quad -(V_\tau \chi_+ W^0(g)\chi_-[\chi_{(-\tau,0)}I - \chi_- I]V_{-\tau}). \qquad (6.5)
\end{aligned}$$

The operators $\chi_- W^0(g)\chi_+ I$ and $\chi_+ W^0(g)\chi_- I$ are compact by Lemma 2.5. It is easy to see that $\chi_{(-\tau,0)}I$ is *-strongly convergent to $\chi_- I$. Thus the sequence on the right hand side of (6.5) has the form $(V_\tau K_1 V_{-\tau}) + (G_\tau^{(1)})$, where $K_1 \in \mathcal{K}$ and $(G_\tau^{(1)}) \in \mathcal{G}$. This proves (6.3). Analogously, it can be shown that

$$(P_\tau \chi_- W^0(g)\chi_- Q_\tau - Q_\tau \chi_- W^0(g)\chi_- P_\tau) = (V_{-\tau}K_{-1}V_\tau) + (G_\tau^{(2)}),$$

where $K_{-1} \in \mathcal{K}$ and $(G_\tau^{(2)}) \in \mathcal{G}$. This proves (6.4) and finishes the proof of the theorem. □

6.4. Central subalgebra of the algebra $\mathcal{L}/\mathcal{J}$ and its maximal ideal space

Let $\mathcal{C}$ be the smallest closed subalgebra of $\mathcal{L}$ that contains all sequences (fI) with $f \in C(\dot{\mathbb{R}})$ and $(W^0(g))$ with $g \in SO_p$. From the proof of Theorem 6.4 it follows that $\mathcal{C}$ is not trivial. Put

$$\mathcal{C}^{\mathcal{J}} := (\mathcal{C} + \mathcal{J})/\mathcal{J}, \quad \mathcal{L}^{\mathcal{J}} := \mathcal{L}/\mathcal{J}.$$

Lemma 6.5. *The set $\mathcal{C}^{\mathcal{J}}$ is a closed central subalgebra of the algebra $\mathcal{L}^{\mathcal{J}}$.*

Proof. This fact follows immediately from the definition of the algebras $\mathcal{L}$, $\mathcal{L}^{\mathcal{J}}$, and $\mathcal{C}^{\mathcal{J}}$. □

Theorem 6.6. *The maximal ideal space $M(\mathcal{C}^{\mathcal{J}})$ of the commutative Banach algebra $\mathcal{C}^{\mathcal{J}}$ is homeomorphic to the set Ω given by* (2.8)*.*

Proof. Since $(V_{\pm\tau})$ converge weakly to zero as $\tau \to +\infty$, it is easy to see that $\mathsf{W}_0(\mathbf{J}) \in \mathcal{K}$ for every $\mathbf{J} \in \mathcal{J}$. Hence

$$\Phi : \mathcal{F}/\mathcal{J} \to \mathcal{B}/\mathcal{K}, \quad \mathbf{A} + \mathcal{J} \mapsto \mathsf{W}_0(\mathbf{A}) + \mathcal{K}$$

is a well defined homomorphism. Clearly $\Phi(\mathcal{J}) = \mathcal{K}$ and for $f \in C(\dot{\mathbb{R}})$ and $g \in SO_p$,

$$\Phi((fI) + \mathcal{J}) = fI + \mathcal{K}, \quad \Phi((W^0(g)) + \mathcal{J}) = W^0(g) + \mathcal{K}.$$

Hence the mapping $\Phi|_{\mathcal{C}^{\mathcal{J}}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^\pi$ is injective and onto, where the algebra $\mathcal{C}^\pi$ is defined just before Lemma 2.9. Therefore $\mathcal{C}^{\mathcal{J}}$ and $\mathcal{C}^\pi$ are isomorphic. Then the maximal ideal spaces of $\mathcal{C}^{\mathcal{J}}$ and $\mathcal{C}^\pi$ are homeomorphic. Thus, from Lemma 2.9 it follows that $M(\mathcal{C}^{\mathcal{J}})$ is homeomorphic to the set Ω given by (2.8). □

6.5. Localization via the Allan-Douglas local principle

With every point $(\xi, \eta) \in \Omega$ we associate the closed two-sided ideal $\mathcal{N}_{\xi,\eta}^{\mathcal{J}}$ of the algebra $\mathcal{C}^{\mathcal{J}}$ generated by all the cosets

$$(fI) + \mathcal{J} \quad (f \in C(\dot{\mathbb{R}}),\ f(\xi) = 0), \qquad (W^0(g)) + \mathcal{J} \quad (g \in SO_p,\ g(\eta) = 0).$$

Let $\mathcal{I}^{\mathcal{J}}_{\xi,\eta}$ be the smallest closed two-sided ideal of $\mathcal{L}^{\mathcal{J}}$ that contains the maximal ideal $\mathcal{N}^{\mathcal{J}}_{\xi,\eta}$ and let

$$\Phi^{\mathcal{J}}_{\xi,\eta} : \mathcal{L}^{\mathcal{J}} \to \mathcal{L}^{\mathcal{J}}/\mathcal{I}^{\mathcal{J}}_{\xi,\eta} =: \mathcal{L}^{\mathcal{J}}_{\xi,\eta}$$

be the canonical homomorphism of $\mathcal{L}^{\mathcal{J}}$ onto the quotient algebra $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$. Now we are in a position to apply the Allan-Douglas local principle. Summarizing the results obtained so far we arrive at the following.

Theorem 6.7. *A sequence* $\mathbf{A} = (A_\tau) \in \mathcal{L}$ *is stable if and only if the operators* $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$, *and* $\mathsf{W}_1(\mathbf{A})$ *are invertible in* $\mathcal{B}$ *and the coset* $\mathbf{A}+\mathcal{I}^{\mathcal{J}}_{\xi,\eta}$ *is invertible in the local algebra* $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$ *for every* $(\xi, \eta) \in \Omega$, *where* Ω *is given by* (2.8).

Proof. From Lemma 5.1(a) and the definition of $\mathcal{L}$ we know that $\mathcal{L} \subset \mathcal{F} \subset \mathcal{E}$. In view of Theorem 4.5, $\mathbf{A}$ is stable if and only if $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{E}/\mathcal{G}$. By Lemma 5.1(e), the latter is equivalent to the invertibility of $\mathbf{A} + \mathcal{G}$ in $\mathcal{F}/\mathcal{G}$. By the lifting theorem (Theorem 5.5), this is equivalent to the invertibility of the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$, and $\mathsf{W}_1(\mathbf{A})$ in the algebra $\mathcal{B}$ and the invertibility of the coset $\mathbf{A} + \mathcal{J}$ in the quotient algebra $\mathcal{F}/\mathcal{J}$. From Lemma 6.2(c) we conclude that $\mathbf{A} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$ if and only if it is invertible in $\mathcal{L}/\mathcal{J}$. The latter assertion is equivalent to the invertibility of the cosets $\mathbf{A}+\mathcal{I}^{\mathcal{J}}_{\xi,\eta}$ in the local algebras $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$ for all $(\xi, \eta) \in \Omega$ by the Allan-Douglas local principle (Theorem 6.1). $\square$

Our next aim is to study the invertibility of the elements

$$\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A}) := \mathbf{A} + \mathcal{I}^{\mathcal{J}}_{\xi,\eta} \quad (\mathbf{A} \in \mathcal{L})$$

in the local algebras $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$. It turns out that the algebras $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$ are too large in order to obtain via Theorem 6.7 effectively verifiable stability conditions for an arbitrary $\mathbf{A} \in \mathcal{L}$. So we restrict ourselves to the case of sequences belonging to $\mathcal{A}$ (recall that $\mathcal{A} \subset \mathcal{L}$ by Theorem 6.4). We denote by $\mathcal{A}^{\mathcal{J}}$ the smallest closed subalgebra of $\mathcal{L}^{\mathcal{J}}$ that contains the cosets $\mathbf{A} + \mathcal{J}$ for all $\mathbf{A} = (A_\tau) \in \mathcal{A}$. The canonical homomorphism $\Phi^{\mathcal{J}}_{\xi,\eta}$ sends $\mathcal{A}^{\mathcal{J}}$ onto

$$\mathcal{A}^{\mathcal{J}}_{\xi,\eta} := \Phi^{\mathcal{J}}_{\xi,\eta}(\mathcal{A}^{\mathcal{J}}) \subset \mathcal{L}^{\mathcal{J}}_{\xi,\eta}.$$

In Sections 8–10 we obtain sufficient conditions for the invertibility of the elements $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A})$ in the local algebras $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$ for $\mathbf{A} \in \mathcal{A}$ and $(\xi, \eta) \in \Omega$.

7. Homogenization

7.1. Algebra of sequences generated by homogeneous operators and (P_τ)

Given any positive real number τ, define the operator

$$Z_\tau : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \quad (Z_\tau f)(x) = \tau^{-1/p} f(x/\tau).$$

Obviously $\|Z_\tau\|_{\mathcal{B}} = 1$. It is also clear that Z_τ is invertible and $Z_\tau^{-1} = Z_{1/\tau}$. An operator $A \in \mathcal{B}$ is called *homogeneous* (or *dilation invariant*) if $Z_\tau^{-1} A Z_\tau = A$ for

each $\tau \in \mathbb{R}_+$. It is easy to see that the operators $\chi_\pm I$ and $W^0(\chi_\pm)$ are homogeneous operators.

Let $\mathcal{H}$ be the smallest closed subalgebra of $\mathcal{E}$ that contains the sequence (P_τ) and the constant sequences (H), where H is a homogeneous operator.

Lemma 7.1. *Let $\mathbf{A} = (A_\tau) \in \mathcal{H} \cap \mathcal{L}$. Then the coset $\mathbf{A} + \mathcal{G}$ is invertible in the algebra $\mathcal{L}/\mathcal{G}$ if and only if the operator A_1 is invertible.*

Proof. The idea of the proof is borrowed from [17, Proposition 1].

Necessity. If $\mathbf{A}+\mathcal{G}$ is invertible in $\mathcal{L}/\mathcal{G}$, then obviously it is invertible in $\mathcal{E}/\mathcal{G}$. Therefore, by Theorem 4.5, $\mathbf{A}$ is stable. Hence the operators A_τ are invertible for all sufficiently large τ. Since

$$Z_\tau^{-1} A_\tau Z_\tau = A_1 \quad (\tau \in \mathbb{R}_+), \tag{7.1}$$

we conclude that the operator A_1 is invertible. The necessity portion is proved.

Sufficiency. If A_1 is invertible, then (7.1) implies that $A_\tau^{-1} = Z_\tau A_1^{-1} Z_\tau^{-1}$. Since $\|Z_\tau\|_{\mathcal{B}} \|Z_\tau^{-1}\|_{\mathcal{B}} = 1$, the norms of A_τ^{-1} are uniformly bounded. That is $\mathbf{A} = (A_\tau)$ is stable. By Theorem 4.5, $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{E}/\mathcal{G}$. But $\mathcal{L}/\mathcal{G}$ is inverse closed in $\mathcal{F}/\mathcal{G}$ (Lemma 6.2(c)) and $\mathcal{F}/\mathcal{G}$ is inverse closed in $\mathcal{E}/\mathcal{G}$ (Lemma 5.1(e)). Thus $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{L}/\mathcal{G}$. □

7.2. Algebras $\mathcal{H}_\eta$ and their ideal $\mathcal{G}$

For $\eta \in \mathbb{R}$, put

$$U_\eta : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \quad (U_\eta f)(x) = e^{i\eta x} f(x).$$

It is clear that $U_\eta^{-1} = U_{-\eta}$ and $\|U_\eta^{\pm 1}\|_{\mathcal{B}} = 1$.

Let $\mathcal{H}_\eta$ denote the set of all sequences $\mathbf{A} = (A_\tau) \in \mathcal{E}$ such that the sequence $(Z_\tau^{-1} U_\eta A_\tau U_\eta^{-1} Z_\tau)$ is *-strongly convergent as $\tau \to +\infty$.

Lemma 7.2. *Suppose $\eta \in \mathbb{R}$.*

(a) *The set $\mathcal{H}_\eta$ is a closed unital subalgebra of the algebra $\mathcal{E}$.*

(b) *The mapping $\mathsf{H}_\eta : \mathcal{H}_\eta \to \mathcal{B}$ given by*

$$\mathsf{H}_\eta(\mathbf{A}) := \operatorname*{s-lim}_{\tau \to +\infty} Z_\tau^{-1} U_\eta A_\tau U_\eta^{-1} Z_\tau$$

for $\mathbf{A} = (A_\tau) \in \mathcal{H}_\eta$ is a bounded unital homomorphism with the norm

$$\|\mathsf{H}_\eta\| = 1.$$

(c) *The set $\mathcal{G}$ is a closed two-sided ideal of the algebra $\mathcal{H}_\eta$.*

(d) *The ideal $\mathcal{G}$ lies in the kernel of the homomorphism H_η.*

(e) *The algebra $\mathcal{H}_\eta/\mathcal{G}$ is inverse closed in the algebra $\mathcal{E}/\mathcal{G}$.*

The proof is analogous to the proof of [20, Proposition 4.1].

Remark 7.3. Let $\widetilde{\mathcal{F}}$ be the algebra denoted by $\mathcal{F}$ in the paper [20]. Then

$$\widetilde{\mathcal{F}} \subset \mathcal{F} \cap \left(\cap_{\eta \in \mathbb{R}} \mathcal{H}_\eta\right).$$

We conjecture that this inclusion is proper.

7.3. The algebra $\mathcal{A}$ is contained in the algebras $\mathcal{H}_\eta$

Proposition 7.4. *Suppose* $\eta \in \mathbb{R}$.

(a) *If* $\mathbf{P} = (P_\tau)$, *then* $\mathbf{P} \in \mathcal{H}_\eta$ *and* $\mathsf{H}_\eta(\mathbf{P}) = P_1$.
(b) *If* $\mathbf{A} = (aI)$ *with* $a \in PC$, *then* $\mathbf{A} \in \mathcal{H}_\eta$ *and*

$$\mathsf{H}_\eta(\mathbf{A}) = a(-\infty)\chi_- I + a(+\infty)\chi_+ I.$$

(c) *If* $\mathbf{B} = (W^0(b))$ *with* $b \in [PC_p, SO_p]$, *then* $\mathbf{B} \in \mathcal{H}_\eta$ *and*

$$\mathsf{H}_\eta(\mathbf{B}) = b(\eta^-)W^0(\chi_-) + b(\eta^+)W^0(\chi_+).$$

Proof. (a) It is easy to see that for every $\tau \in \mathbb{R}_+$ one has $U_\eta P_\tau U_\eta^{-1} = P_\tau$ and $Z_\tau^{-1} P_\tau Z_\tau = P_1$. From these equalities it follows immediately that $\mathsf{H}_\eta(\mathbf{P}) = P_1$. Taking into account that

$$Z_\tau^* = Z_{1/\tau}, \quad U_\eta^* = U_{-\eta}, \quad P_\tau^* = P_\tau,$$

from above we conclude that $(Z_\tau^{-1} U_\eta P_\tau U_\eta^{-1} Z_\tau)^*$ converges strongly to P_1^*. Thus $\mathbf{P} \in \mathcal{H}_\eta$. Part (a) is proved.

Part (b) is proved in [21, Proposition 13.1(b)] and the proof of part (c) can be developed by analogy with [2, Theorem 4.2(i)]. □

Remark 7.5. For $\mathbf{A} \in \mathcal{A}$, each operator $\mathsf{H}_\eta(\mathbf{A})$ is homogeneous. So the passage from a sequence $\mathbf{A} \in \mathcal{A}$ to its image under the homomorphism H_η can be naturally called *homogenization*.

Corollary 7.6. *For every* $\eta \in \mathbb{R}$, *the algebra* $\mathcal{A}$ *is a closed unital subalgebra of the algebra* $\mathcal{H}_\eta$.

7.4. Necessary condition for the stability of $\mathbf{A} \in \mathcal{A}$

Now we are in a position to prove the main result of this section.

Theorem 7.7. *If a sequence* $\mathbf{A} \in \mathcal{A}$ *is stable, then the operators* $\mathsf{H}_\eta(\mathbf{A})$ *are invertible in the algebra* $\mathcal{B}$ *for all* $\eta \in \mathbb{R}$.

Proof. If $\mathbf{A} \in \mathcal{A}$ is stable, then by Theorem 4.5, the coset $\mathbf{A} + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{E}/\mathcal{G}$. Then from Lemma 7.2(e) it follows that $\mathbf{A} + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{H}_\eta/\mathcal{G}$ for every $\eta \in \mathbb{R}$. This means that there exist $\mathbf{B} \in \mathcal{H}_\eta$ and $\mathbf{G}_1, \mathbf{G}_2 \in \mathcal{G}$ such that

$$\mathbf{AB} = \mathbf{I} + \mathbf{G}_1, \quad \mathbf{BA} = \mathbf{I} + \mathbf{G}_2. \tag{7.2}$$

From Lemma 7.2(b),(d) we know that $\mathsf{H}_\eta : \mathcal{H}_\eta \to \mathcal{B}$ is a unital homomorphism and $\mathsf{H}_\eta(\mathbf{G}_1) = \mathsf{H}_\eta(\mathbf{G}_2) = 0$. Applying this homomorphism to the equalities in (7.2), we obtain that $\mathsf{H}_\eta(\mathbf{A})$ is invertible in $\mathcal{B}$ and its inverse is equal to $\mathsf{H}_\eta(\mathbf{B})$. □

Remark 7.8. The proof of Theorem 7.7 given above does not use the results of Sections 5 and 6.

7.5. Strong convergence of families of sequences associated to the fiber $M_\infty(SO)$

The following statement can be considered as a counterpart of Proposition 7.4(c) for the fiber $M_\infty(SO)$. It will be used in Section 10 and can be proved by analogy with [2, Theorem 4.2(ii)] with the aid of Lemma 2.8.

Lemma 7.9. *Suppose $\eta \in M_\infty(SO)$ and $g_1, \dots, g_m \in SO_p$ is a finite family such that $\eta(g_1) = \dots = \eta(g_m) = 0$. Then there exists a sequence $(\tau_n)_{n=1}^\infty \subset \mathbb{R}_+$ such that $\tau_n \to +\infty$ as $n \to \infty$ and*

$$\operatorname*{s-lim}_{n\to\infty} Z_{\tau_n} W^0(g_j) Z_{\tau_n}^{-1} = 0 \quad \textit{for all} \quad j = 1, \dots, m.$$

8. Invertibility in the local algebra $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$ with $(\xi, \eta) \in \mathbb{R} \times M_\infty(SO)$

8.1. Local images of elements of $\mathcal{A}$

Lemma 8.1. *Suppose $\mathbf{A} \in \mathcal{A}$ and $(\xi, \eta) \in \mathbb{R} \times M_\infty(SO)$. Then the constant sequence $\mathbf{A}_0 := (\mathsf{W}_0(\mathbf{A}))$ belongs to $\mathcal{A}$ and*

$$\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A}) = \Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A}_0). \tag{8.1}$$

Proof. This statement is proved by analogy with [18, Proposition 2.14]. If $\mathbf{A} = (aI)$ with $a \in PC$ or $\mathbf{A} = (W^0(b))$ with $b \in [PC_p, SO_p]$, then obviously $\mathbf{A}_0 = \mathbf{A} \in \mathcal{A}$. For $\mathbf{P} = (P_\tau) \in \mathcal{A}$, we have $\mathsf{W}_0(\mathbf{P}) = I$ and $\mathbf{P}_0 = (I) =: \mathbf{I} \in \mathcal{A}$. These facts imply that $\mathbf{A}_0 \in \mathcal{A}$.

For all constant sequences $\mathbf{A} \in \mathcal{A}$, we have $\mathbf{A} = \mathbf{A}_0 \in \mathcal{A}$. This implies (8.1) for such sequences. It remains to show that $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{P}) = \Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{I})$.

Assume that $y > |\xi|$ and consider a function $f_\xi \in C(\dot{\mathbb{R}})$ such that $f_\xi(\xi) = 1$ and $\operatorname{supp} f_\xi \subset (-y, y)$. From the definition of the ideal $\mathcal{I}^{\mathcal{J}}_{\xi,\eta}$ it follows that

$$(f_\xi I) - (I) + \mathcal{J} = ((f_\xi - 1)I) + \mathcal{J} \in \mathcal{I}^{\mathcal{J}}_{\xi,\eta}.$$

Hence $\Phi^{\mathcal{J}}_{\xi,\eta}[(f_\xi I)] = \Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{I})$ is the identity in the local algebra $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$. Put $\mathbf{Q} := (Q_\tau)$. Then

$$\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{Q}) = \Phi^{\mathcal{J}}_{\xi,\eta}[(f_\xi I)] \Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{Q}) = \Phi^{\mathcal{J}}_{\xi,\eta}[(f_\xi Q_\tau)].$$

Since $\operatorname{supp} f_\xi \subset (-y, y)$, we have $\|f_\xi Q_\tau\|_{\mathcal{B}} \to 0$ as $\tau \to \infty$. Then $(f_\xi Q_\tau) \in \mathcal{G} \subset \mathcal{J}$. This means that $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{Q})$ is the zero in $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$. Hence $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{P}) = \Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{I})$. □

8.2. Sufficient conditions for the invertibility in $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$

Theorem 8.2. *Let $(\xi, \eta) \in \mathbb{R} \times M_\infty(SO)$. If $\mathbf{A} \in \mathcal{A}$ and the operator $\mathsf{W}_0(\mathbf{A})$ is invertible, then the coset $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A})$ is invertible in the local algebra $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$.*

Proof. If $\mathsf{W}_0(\mathbf{A})$ is invertible, then the constant sequence $\mathbf{A}_0 = (\mathsf{W}_0(\mathbf{A})) \in \mathcal{A}$ is stable. From Theorem 4.5 and Lemma 5.1(e) we obtain that $\mathbf{A}_0 + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$. Therefore, $\mathbf{A}_0 + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{J}$ because $\mathcal{G} \subset \mathcal{J}$. From Lemma 6.2(c) it follows that the latter fact is equivalent to the invertibility

of $\mathbf{A}_0+\mathcal{J}$ in the quotient algebra $\mathcal{L}/\mathcal{J}$. By Theorem 6.1, this implies that $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A}_0)$ is invertible in the local algebra $\mathcal{L}^{\mathcal{J}}_{\xi,\eta}$. It remains to recall that $\Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A}) = \Phi^{\mathcal{J}}_{\xi,\eta}(\mathbf{A}_0)$ by Lemma 8.1(b). □

9. Invertibility in the local algebra $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ with $\eta \in \mathbb{R}$

9.1. Two auxiliary lemmas

Given $\eta \in \mathbb{R}$, let χ^-_η and χ^+_η denote the characteristic functions of $(-\infty,\eta)$ and $(\eta,+\infty)$, respectively. Clearly, $\chi^-_0 = \chi_-$ and $\chi^+_0 = \chi_+$. It is easy to see that

$$W^0(\chi^-_\eta) = U^{-1}_\eta W^0(\chi_-)U_\eta, \quad W^0(\chi^+_\eta) = U^{-1}_\eta W^0(\chi_+)U_\eta.$$

Consider the constant sequences

$$\mathbf{X}_- := (\chi_- I), \quad \mathbf{X}_+ := (\chi_+ I), \quad \mathbf{W}^\eta_- := (W^0(\chi^-_\eta)), \quad \mathbf{W}^\eta_+ := (W^0(\chi^+_\eta)).$$

Lemma 9.1. *Let $\eta \in \mathbb{R}$.*

(a) *If $a \in PC$ and $\mathbf{A} = (aI)$, then*

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}) = a(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) + a(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+).$$

(b) *If $b \in [PC_p, SO_p]$ and $\mathbf{B} = (W^0(b))$, then*

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{B}) = b(\eta^-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}^\eta_-) + b(\eta^+)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}^\eta_+).$$

Proof. (a) Consider $a_0 := a - [a(-\infty)\chi_- + a(+\infty)\chi_+]$. Clearly $a_0 \in PC$ and $a_0(-\infty) = a_0(+\infty) = 0$. Therefore $(a_0 I) + \mathcal{J} \in \mathcal{I}^{\mathcal{J}}_{\infty,\eta}$. Hence

$$0 = \Phi^{\mathcal{J}}_{\infty,\eta}[(a_0 I)] = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}) - [a(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) + a(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)].$$

Part (a) is proved.

(b) The function $b_0 := b - [b(\eta^-)\chi^-_\eta + b(\eta^+)\chi^+_\eta]$ belongs to $[PC_p, SO_p]$ and, moreover, it is continuous and vanishing at the point $\eta \in \mathbb{R}$. Hence the coset $(W^0(b_0)) + \mathcal{J}$ belongs to the ideal $\mathcal{I}^{\mathcal{J}}_{\infty,\eta}$. Therefore

$$0 = \Phi^{\mathcal{J}}_{\infty,\eta}[(W^0(b_0 I))] = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{B}) - [b(\eta^-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}^\eta_-) + b(\eta^+)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}^\eta_+)].$$

This completes the proof of part (b). □

Lemma 9.2. *Suppose $\mathbf{A} \in \mathcal{A}$ and $\eta \in \mathbb{R}$. Then the sequence $\mathbf{A}_\eta$ given by*

$$\mathbf{A}_\eta := (U^{-1}_\eta Z_\tau \mathsf{H}_\eta(\mathbf{A}) Z^{-1}_\tau U_\eta)$$

belongs to $\mathcal{A}$ and

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}_\eta). \tag{9.1}$$

Proof. From Proposition 7.4(a) it follows that

$$U^{-1}_\eta Z_\tau \mathsf{H}_\eta(\mathbf{P}) Z^{-1}_\tau U_\eta = U^{-1}_\eta Z_\tau P_1 Z^{-1}_\tau U_\eta = P_\tau. \tag{9.2}$$

By Proposition 7.4(b), for $\mathbf{A} = (aI)$ with $a \in PC$ we have

$$\begin{aligned} U_\eta^{-1} Z_\tau \mathsf{H}_\eta(\mathbf{A}) Z_\tau^{-1} U_\eta &= U_\eta^{-1} Z_\tau [a(-\infty)\chi_- I + a(+\infty)\chi_+ I] Z_\tau^{-1} U_\eta \\ &= a(-\infty) Z_\tau \chi_- Z_\tau^{-1} + a(+\infty) Z_\tau \chi_+ Z_\tau^{-1} \\ &= a(-\infty)\chi_- I + a(+\infty)\chi_+ I. \end{aligned} \tag{9.3}$$

Assume that $b \in [PC_p, SO_p]$ and $\mathbf{A} = (W^0(b))$. By Proposition 7.4(c),

$$\begin{aligned} & U_\eta^{-1} Z_\tau \mathsf{H}_\eta(\mathbf{A}) Z_\tau^{-1} U_\eta = U_\eta^{-1} Z_\tau [b(\eta^-) W^0(\chi_-) + b(\eta^+) W^0(\chi_+)] Z_\tau^{-1} U_\eta \\ & = b(\eta^-) U_\eta^{-1} Z_\tau W^0(\chi_-) Z_\tau^{-1} U_\eta + b(\eta^+) U_\eta^{-1} Z_\tau W^0(\chi_+) Z_\tau^{-1} U_\eta \\ & = b(\eta^-) U_\eta^{-1} W^0(\chi_-) U_\eta + b(\eta^+) U_\eta^{-1} W^0(\chi_+) U_\eta \\ & = b(\eta^-) W^0(\chi_\eta^-) + b(\eta^+) W^0(\chi_\eta^+). \end{aligned} \tag{9.4}$$

Equalities (9.2)–(9.4) imply that $\mathbf{A}_\eta \in \mathcal{A}$ for every $\mathbf{A} \in \mathcal{A}$.

Obviously, it is sufficient to check identity (9.1) on the generators of the algebra $\mathcal{A}$. From (9.2) we get

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}_\eta) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})$$

for $\mathbf{A} = \mathbf{P}$. From Lemma 9.1(a) and equality (9.3) it follows that

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}_\eta) = a(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) + a(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})$$

for $\mathbf{A} = (aI)$ with $a \in PC$. Equality (9.4) and Lemma 9.1(b) yield

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}_\eta) = b(\eta^-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}^\eta_-) + b(\eta^+)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}^\eta_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})$$

with $\mathbf{A} = (W^0(b))$ and $b \in [PC_p, SO_p]$. □

9.2. Sufficient condition for the invertibility in $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$

Theorem 9.3. *Let $\eta \in \mathbb{R}$ and $\mathbf{A} \in \mathcal{A}$. If the operator $\mathsf{H}_\eta(\mathbf{A})$ is invertible in the algebra $\mathcal{B}$, then the coset $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})$ is invertible in the local algebra $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$.*

Proof. Let $B \in \mathcal{B}$ be the inverse of $\mathsf{H}_\eta(\mathbf{A})$. Multiplying the equalities

$$B\mathsf{H}_\eta(\mathbf{A}) = I = \mathsf{H}_\eta(\mathbf{A})B$$

from the left by $U_\eta^{-1} Z_\tau$ and from the right by $Z_\tau^{-1} U_\eta$, we obtain

$$\mathbf{B}_{(\eta)}\mathbf{A}_\eta = \mathbf{I} = \mathbf{A}_\eta \mathbf{B}_{(\eta)},$$

where $\mathbf{B}_{(\eta)} := (U_\eta^{-1} Z_\tau B Z_\tau^{-1} U_\eta) \in \mathcal{E}$. It is easy to see that $\|\mathbf{B}_{(\eta)}\|_{\mathcal{E}} \le \|B\|_{\mathcal{B}}$. Hence all operators $U_\eta^{-1} Z_\tau \mathsf{H}_\eta(\mathbf{A}) Z_\tau^{-1} U_\eta$ are invertible and the norms of their inverses are uniformly bounded. Thus the sequence $\mathbf{A}_\eta$ is stable. From Lemma 9.2 and Theorem 6.4 we conclude that $\mathbf{A}_\eta \in \mathcal{L}$. Then the stability of $\mathbf{A}_\eta$ implies the invertibility of the coset $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}_\eta)$ in the local algebra $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ due to Theorem 6.7. It remains to recall that $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}_\eta) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})$ by Lemma 9.2. □

10. Invertibility in the local algebra $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ with $\eta \in M_\infty(SO)$

10.1. On the center of the algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$

Let $\mathbf{P} := (P_\tau)$ and $\mathbf{Q} := (Q_\tau)$. From Lemma 2.3 it follows that the functions

$$u_\pm(x) := (1 \pm \tanh x)/2 \quad (x \in \overline{\mathbb{R}})$$

belong to $C_p(\overline{\mathbb{R}})$. Hence the convolution operators $W^0(u_\pm)$ are well defined and bounded on $L^p(\mathbb{R})$. Consider the following constant sequences

$$\mathbf{U}_- := (u_- I), \quad \mathbf{U}_+ := (u_+ I), \quad \mathbf{V}_- := (W^0(u_-)), \quad \mathbf{V}_+ := (W^0(u_+)),$$
$$\mathbf{X}_- := (\chi_- I), \quad \mathbf{X}_+ := (\chi_+ I), \quad \mathbf{W}_- := (W^0(\chi_-)), \quad \mathbf{W}_+ := (W^0(\chi_+)).$$

Lemma 10.1. *Let $\eta \in M_\infty(SO)$.*

(a) *We have*

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_-), \quad \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+),$$
$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_-), \quad \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_+).$$

(b) *If $a \in PC$ and $\mathbf{A} = (aI)$, then*

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}) = a(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) + a(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+).$$

(c) *If $b \in [PC_p, SO_p]$ and $\mathbf{B} = (W^0(b))$, then*

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{B}) = b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_-) + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_+).$$

Proof. (a) Since $\chi_-(-\infty) = u_-(-\infty)$ and $\chi_-(+\infty) = u_-(+\infty)$, the function $\chi_- - u_- \in PC_p$ is continuous and vanishing at the point ∞ of $\dot{\mathbb{R}}$. Hence the cosets $\mathbf{X}_- - \mathbf{U}_- + \mathcal{J}$ and $\mathbf{W}_- - \mathbf{V}_- + \mathcal{J}$ belong to the ideal $\mathcal{I}^{\mathcal{J}}_{\infty,\eta}$. Therefore $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_-)$ and $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_-)$. Analogously one can show that $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+)$ and $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_+)$.

(b) The proof coincides with the proof of Lemma 9.1(a).

(c) Consider the function $\widetilde{b} := b - b_\eta(-\infty)\chi_- - b_\eta(+\infty)\chi_+ \in [PC_p, SO_p]$. Then $(\alpha_\eta \widetilde{b})(-\infty) = (\alpha_\eta \widetilde{b})(+\infty) = 0$. In particular, if $b \in SO_p$, then $\widetilde{b} = b - b(\eta)$ and $\widetilde{b}(\eta) = 0$. This implies that $(W^0(\widetilde{b})) + \mathcal{J} \in \mathcal{I}^{\mathcal{J}}_{\infty,\eta}$ for $b \in [PC_p, SO_p]$. Thus

$$0 = \Phi^{\mathcal{J}}_{\infty,\eta}[(W^0(\widetilde{b}))] = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{B}) - b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) - b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+),$$

which finishes the proof of part (c). □

Lemma 10.2. *Let $\eta \in M_\infty(SO)$. Then the cosets $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)$ and $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)$ commute with any element of the local algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$.*

Proof. We will prove that $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)$ commutes with the elements of $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$. It is sufficient to prove this statement for the generators of $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$. It is obvious that

$\chi_+ P_\tau = P_\tau \chi_+ I$ and $\chi_+ a = a\chi_+$ for all $a \in PC$. As usual, denote $\mathbf{P} = (P_\tau)$ and $\mathbf{A} = (aI)$. Then

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P})\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+),$$
$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A}) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+).$$

By Lemma 10.1(a),(c),

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+), \tag{10.1}$$
$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{B}) = b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_-) + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_+). \tag{10.2}$$

In view of Lemma 2.6, $u_+W^0(u_-) - W^0(u_-)u_+I \in \mathcal{K}$. Hence $\mathbf{U}_+\mathbf{V}_- - \mathbf{V}_-\mathbf{U}_+ \in \mathcal{J}$ and therefore

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+\mathbf{V}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_-\mathbf{U}_+). \tag{10.3}$$

Analogously,

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+\mathbf{V}_+) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_+\mathbf{U}_+). \tag{10.4}$$

Combining (10.1)–(10.4), we get

$$\begin{aligned}
\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{B}) &= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+)\big[b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_-) + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_+)\big] \\
&= b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+\mathbf{V}_-) + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+\mathbf{V}_+) \\
&= b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_-\mathbf{U}_+) + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_+\mathbf{U}_+) \\
&= \big[b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_-) + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{V}_+)\big]\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{U}_+) \\
&= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{B})\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+),
\end{aligned}$$

which finishes the proof of the statement for $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)$. This immediately implies that $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)$ commutes with the elements of $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$ because $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{I}) - \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)$ and $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{I})$ is the identity of $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$. □

10.2. Reduction to algebras generated by two idempotents

An element p of a Banach algebra B is said to be *idempotent* if $p^2 = p$. If B is a unital Banach algebra with identity e, then $e - p$ is also an idempotent and

$$pBp := \big\{pap : a \in B\big\}, \quad (e-p)B(e-p) := \big\{(e-p)a(e-p) : a \in B\big\}$$

are unital Banach algebras with the identities p and $e - p$, respectively.

Lemma 10.3. *Let B be a Banach algebra with identity e and let $p \neq e$ be an idempotent element of B. Suppose A is a closed subalgebra of B that contains e and p. If p commutes with any element of A, then*

(a) *an element $a \in A$ is invertible in the algebra B if and only if the element pap is invertible in the algebra pBp and the element $(e-p)a(e-p)$ is invertible in the algebra $(e-p)B(e-p)$;*

(b) *the algebra A is inverse closed in the algebra B if and only if the algebra pAp is inverse closed in the algebra pBp and the algebra $(e-p)A(e-p)$ is inverse closed in the algebra $(e-p)B(e-p)$.*

The proof of this lemma is straightforward and therefore it is omitted.

The above results allow us to split the algebras $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$ and $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ into pairs of simpler algebras $\mathcal{A}^{\pm}_{\eta}$ and $\mathcal{L}^{\pm}_{\eta}$. The algebras $\mathcal{A}^{-}_{\eta}$ and $\mathcal{A}^{+}_{\eta}$ have a nice algebraic structure: they are generated by two idempotents and the identity.

Lemma 10.4. *Let $\eta \in M_\infty(SO)$ and*

$$\mathcal{A}^{-}_{\eta} := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\mathcal{A}^{\mathcal{J}}_{\infty,\eta}\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-), \quad \mathcal{A}^{+}_{\eta} := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)\mathcal{A}^{\mathcal{J}}_{\infty,\eta}\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+),$$
$$\mathcal{L}^{-}_{\eta} := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\mathcal{L}^{\mathcal{J}}_{\infty,\eta}\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-), \quad \mathcal{L}^{+}_{\eta} := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)\mathcal{L}^{\mathcal{J}}_{\infty,\eta}\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+).$$

(a) *The elements*

$$p_- := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-), \quad r_- := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_-\mathbf{X}_-)$$

are idempotents in the algebra $\mathcal{L}^{-}_{\eta}$ and the algebra $\mathcal{A}^{-}_{\eta}$ is the smallest closed subalgebra of $\mathcal{L}^{-}_{\eta}$ that contains p_-, r_-, and the identity $e_- := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)$.

(b) *The elements*

$$p_+ := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_+), \quad r_+ := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_+\mathbf{X}_+)$$

are idempotents in the algebra $\mathcal{L}^{+}_{\eta}$ and the algebra $\mathcal{A}^{+}_{\eta}$ is the smallest closed subalgebra of $\mathcal{L}^{+}_{\eta}$ that contains p_+, r_+, and the identity $e_+ := \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)$.

Proof. (a) Since $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)$ commutes with any element of $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$ by Lemma 10.2 and $\mathbf{X}_-^2 = \mathbf{X}_-$, we have

$$\begin{aligned} p_- &= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-\mathbf{P}\mathbf{X}_-) \\ &= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P})\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) \in \mathcal{A}^{-}_{\eta} \end{aligned} \tag{10.5}$$

and taking into account that $\mathbf{P}^2 = \mathbf{P}$, we have

$$\begin{aligned} p_-^2 &= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-) = \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}^2)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-^2) \\ &= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P})\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) = p_-. \end{aligned}$$

Analogously, taking into account that $\mathbf{W}_-^2 = \mathbf{W}_-$, one can verify that r_- belongs to $\mathcal{A}^{-}_{\eta}$ and $r_-^2 = r_-$.

It remains to show that $\mathcal{A}^{-}_{\eta}$ is generated by p_-, r_-, and e_-. In view of (10.5) it is sufficient to show that $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-\mathbf{A}\mathbf{X}_-)$ and $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-\mathbf{B}\mathbf{X}_-)$, where $\mathbf{A} = (aI)$ with $a \in PC$ and $\mathbf{B} = (W^0(b))$ with $b \in [PC_p, SO_p]$, are represented as linear combinations of p_-, r_-, and e_-.

From Lemmas 10.1 and 10.2 it follows that

$$\begin{aligned} &\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-\mathbf{A}\mathbf{X}_-) \\ &= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\big[a(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) + a(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)\big]\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) \\ &= a(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) \\ &\quad + a(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-) \\ &= a(-\infty)e_- \end{aligned} \tag{10.6}$$

and

$$\begin{aligned}
&\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-\mathbf{B}\mathbf{X}_-)\\
&= \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\big[b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_-) + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_+)\big]\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\\
&= b_\eta(-\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_-)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\\
&\quad + b_\eta(+\infty)\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)[\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{I}) - \Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{W}_-)]\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-)\\
&= b_\eta(-\infty)r_- + b_\eta(+\infty)(e_- - r_-),
\end{aligned} \tag{10.7}$$

which finishes the proof of part (a).

(b) The proof of part (b) is similar. We only note that

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+\mathbf{P}\mathbf{X}_+) = p_+, \tag{10.8}$$

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+\mathbf{A}\mathbf{X}_+) = a(+\infty)e_+, \tag{10.9}$$

$$\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+\mathbf{B}\mathbf{X}_+) = b_\eta(-\infty)(e_+ - r_+) + b_\eta(+\infty)r_+ \tag{10.10}$$

for further references. □

10.3. The two idempotents theorem

The algebras $\mathcal{A}^-_\eta$ and $\mathcal{A}^+_\eta$ introduced in the previous subsection are generated by two idempotent elements and the identity. Invertibility of elements of such algebras can be described with the aid of the so-called two idempotents theorem presented below.

Theorem 10.5 ([3, Theorem 8.7])**.** *Let B be a Banach algebra with identity e and let p and r be idempotents in B. Let further A stand for the smallest closed subalgebra of B containing e, p, and r. Put*

$$X := prp + (e-p)(e-r)(e-p) \tag{10.11}$$

and suppose the points 0 and 1 are cluster points of $\mathrm{sp}_B(X)$. For $x \in \mathbb{C}$, define the map $\sigma_x : \{e, p, r\} \to \mathbb{C}^{2\times 2}$ by

$$\sigma_x(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{10.12}$$

$$\sigma_x(r) = \begin{bmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{bmatrix}, \tag{10.13}$$

where $\sqrt{x(1-x)}$ denotes any complex number such that its square is equal to $x(1-x)$.

(a) *For each $x \in \mathrm{sp}_B(X)$ the map σ_x extends to a Banach algebra homomorphism σ_x of A onto $\mathbb{C}^{2\times 2}$.*

(b) *An element $a \in A$ is invertible in B if and only if $\sigma_x(a)$ is invertible in $\mathbb{C}^{2\times 2}$ for every $x \in \mathrm{sp}_B(X)$.*

(c) *An element $a \in A$ is invertible in A if and only if $\sigma_x(a)$ is invertible in $\mathbb{C}^{2\times 2}$ for every $x \in \mathrm{sp}_A(X)$.*

10.4. Spectra of two elements of the algebra $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$

To apply the two idempotents theorem to the algebras $\mathcal{A}^{\pm}_{\eta}$ and $\mathcal{L}^{\pm}_{\eta}$, we need to find the spectrum of the canonical element given by (10.11) in these algebras. This requires the following auxiliary result.

Lemma 10.6. *Let $\eta \in M_\infty(SO)$ and $\mathfrak{L}_p$ be the lentiform domain defined by* (3.2). *Then*

$$\mathrm{sp}_{\mathcal{L}^{\mathcal{J}}_{\infty,\eta}}\left(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q})\right) = \mathfrak{L}_p, \tag{10.14}$$

$$\mathrm{sp}_{\mathcal{L}^{\mathcal{J}}_{\infty,\eta}}\left(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_+\mathbf{W}_+\mathbf{X}_+\mathbf{P}+\mathbf{Q}\mathbf{X}_+\mathbf{W}_-\mathbf{X}_+\mathbf{Q})\right) = \mathfrak{L}_p. \tag{10.15}$$

Proof. Let us prove equality (10.14). It is easy to see that

$$\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I} \in \mathcal{H}\cap\mathcal{L},$$

where the algebra $\mathcal{H}$ is defined in Subsection 7.1. By Lemma 7.1, the coset

$$\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I}+\mathcal{G}$$

is invertible in the algebra $\mathcal{L}/\mathcal{G}$ if and only if the operator

$$\begin{aligned}&P_1\chi_-W^0(\chi_-)\chi_-P_1+Q_1\chi_-W^0(\chi_+)\chi_-Q_1-\lambda I\\&=\chi_{(-1,0)}P_{\mathbb{R}}\chi_{(-1,0)}I+\chi_{(-\infty,-1)}Q_{\mathbb{R}}\chi_{(-\infty,-1)}I-\lambda I\end{aligned}$$

is invertible on $L^p(\mathbb{R})$. Hence, taking into account Corollary 3.3(a), we obtain

$$\begin{aligned}&\mathrm{sp}_{\mathcal{L}/\mathcal{G}}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}+\mathcal{G})\\&=\mathrm{sp}_{\mathcal{B}}\left(\chi_{(-1,0)}P_{\mathbb{R}}\chi_{(-1,0)}I+\chi_{(-\infty,-1)}Q_{\mathbb{R}}\chi_{(-\infty,-1)}I\right)=\mathfrak{L}_p.\end{aligned}$$

If $\lambda \notin \mathfrak{L}_p$, then there exists a sequence $\mathbf{A}\in\mathcal{L}$ such that

$$\begin{aligned}&(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I}+\mathcal{G})(\mathbf{A}+\mathcal{G})=\mathbf{I}+\mathcal{G},\\&(\mathbf{A}+\mathcal{G})(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I}+\mathcal{G})=\mathbf{I}+\mathcal{G}.\end{aligned}$$

Obviously, the same equalities are true with $\mathcal{G}$ replaced by the larger ideal $\mathcal{J}$. Applying the homomorphism $\Phi^{\mathcal{J}}_{\infty,\eta}:\mathcal{L}^{\mathcal{J}}\to\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ to those equalities, we obtain

$$\begin{aligned}&\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I})\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})=\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{I}),\\&\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I})=\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{I}).\end{aligned}$$

Then $\lambda\notin\mathrm{sp}_{\mathcal{L}^{\mathcal{J}}_{\infty,\eta}}\left(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q})\right)$. Thus

$$\mathrm{sp}_{\mathcal{L}^{\mathcal{J}}_{\infty,\eta}}\left(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q})\right)\subset\mathfrak{L}_p. \tag{10.16}$$

Assume now that

$$\lambda\notin\mathrm{sp}_{\mathcal{L}^{\mathcal{J}}_{\infty,\eta}}\left(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q})\right).$$

Then there exists a sequence $\mathbf{B}\in\mathcal{L}$ such that

$$\begin{aligned}&\mathbf{B}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I})-\mathbf{I}\in\mathcal{I}^{\mathcal{J}}_{\infty,\eta},\\&(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P}+\mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q}-\lambda\mathbf{I})\mathbf{B}-\mathbf{I}\in\mathcal{I}^{\mathcal{J}}_{\infty,\eta},\end{aligned}$$

where $\mathcal{I}^{\mathcal{J}}_{\infty,\eta}$ was defined in Subsection 6.5. From that definition and the above relations it follows that without loss of generality we may assume that there exist finite sums

$$\mathbf{J}_1 + \sum_{i=1}^{n} \mathbf{C}_i(f_i I) + \sum_{j=1}^{m} \mathbf{D}_j(W^0(g_j)) \in \mathcal{I}^{\mathcal{J}}_{\infty,\eta},$$
$$\mathbf{J}_2 + \sum_{i=1}^{k} (h_i I)\mathbf{E}_i + \sum_{j=1}^{l} (W^0(v_j))\mathbf{F}_j \in \mathcal{I}^{\mathcal{J}}_{\infty,\eta}$$

with $\mathbf{C}_i, \mathbf{D}_j, \mathbf{E}_i, \mathbf{F}_j \in \mathcal{L}$ and

$$f_i, h_i \in C(\dot{\mathbb{R}}), \quad f_i(\infty) = h_i(\infty) = 0, \quad g_j, v_j \in SO_p, \quad g_j(\eta) = v_j(\eta) = 0$$

for all i, j and elements $\mathbf{J}_1, \mathbf{J}_2 \in \mathcal{J}$ such that

$$\mathbf{B}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P} + \mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q} - \lambda\mathbf{I}) - \mathbf{I} = \mathbf{J}_1 + \sum_{i=1}^{n} \mathbf{C}_i(f_i I) + \sum_{j=1}^{m} \mathbf{D}_j(W^0(g_j)),$$

$$(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P} + \mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q} - \lambda\mathbf{I})\mathbf{B} - \mathbf{I} = \mathbf{J}_2 + \sum_{i=1}^{k} (h_i I)\mathbf{E}_i + \sum_{j=1}^{l} (W^0(v_j))\mathbf{F}_j.$$

Applying the homomorphism W_{-1} to these equalities, taking into account Proposition 5.2 and that $\mathsf{W}_{-1}(\mathbf{J}_1) = K_1 \in \mathcal{K}$, $\mathsf{W}_{-1}(\mathbf{J}_2) = K_2 \in \mathcal{K}$, we obtain

$$\mathsf{W}_{-1}(\mathbf{B})H_\lambda - I = K_1 + \sum_{j=1}^{m} \mathsf{W}_{-1}(\mathbf{D}_j)W^0(g_j), \tag{10.17}$$

$$H_\lambda \mathsf{W}_{-1}(\mathbf{B}) - I = K_2 + \sum_{j=1}^{l} W^0(v_j)\mathsf{W}_{-1}(\mathbf{F}_j), \tag{10.18}$$

where

$$H_\lambda := \chi_+ W^0(\chi_-)\chi_+ I + \chi_- W^0(\chi_+)\chi_- I - \lambda I = \chi_+ P_{\mathbb{R}} \chi_+ I + \chi_- Q_{\mathbb{R}} \chi_- I - \lambda I.$$

By Lemma 7.9, for $g_1, \dots, g_m$ there exists a sequence $(\tau_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ such that $\tau_n \to +\infty$ and

$$\operatorname*{s-lim}_{n\to\infty} Z_{\tau_n} W^0(g_j) Z_{\tau_n}^{-1} = 0 \quad \text{on} \quad L^p(\mathbb{R}) \tag{10.19}$$

for all $j = 1, \dots, m$. From Lemma 2.4 it follows that $\overline{v_1}, \dots, \overline{v_l} \in SO_q$. So, applying Lemma 7.9 to the space $L^q(\mathbb{R})$, we can guarantee that there exists a sequence $(\tau_n^*)_{n=1}^{\infty} \subset \mathbb{R}_+$ such that $\tau_n^* \to +\infty$ and

$$\operatorname*{s-lim}_{n\to\infty} Z_{\tau_n^*} W^0(\overline{v_j}) Z_{\tau_n^*}^{-1} = 0 \quad \text{on} \quad L^q(\mathbb{R}) \tag{10.20}$$

for all $j = 1, \dots, l$.

Since K_1, K_2 are compact and $(Z_{\tau_n}^{\pm 1})_{n=1}^\infty$ converge weakly to zero on $L^p(\mathbb{R})$ and $(Z_{\tau_n^*}^{\pm 1})_{n=1}^\infty$ converge weakly to zero on $L^q(\mathbb{R})$ as $n \to \infty$, we conclude that

$$\operatorname*{s-lim}_{n\to\infty} Z_{\tau_n} K_1 Z_{\tau_n}^{-1} = 0 \quad \text{on} \quad L^p(\mathbb{R}), \tag{10.21}$$

$$\operatorname*{s-lim}_{n\to\infty} Z_{\tau_n^*} K_2^* Z_{\tau_n^*}^{-1} = 0 \quad \text{on} \quad L^q(\mathbb{R}). \tag{10.22}$$

Multiplying (10.17) from the left by Z_{τ_n} and from the right by $Z_{\tau_n}^{-1}$ and taking into account that the operator H_λ is homogeneous, we obtain

$$\begin{aligned}&(Z_{\tau_n}\mathsf{W}_{-1}(\mathbf{B})Z_{\tau_n}^{-1})H_\lambda - I\\ &= Z_{\tau_n}K_1Z_{\tau_n}^{-1} + \sum_{j=1}^m (Z_{\tau_n}\mathsf{W}_{-1}(\mathbf{D}_j)Z_{\tau_n}^{-1})(Z_{\tau_n}W^0(g_j)Z_{\tau_n}^{-1}) =: A_n.\end{aligned} \tag{10.23}$$

Since $\|Z_{\tau_n}\mathsf{W}_{-1}(\mathbf{D}_j)Z_{\tau_n}^{-1}\|_{\mathcal{B}} \le \|\mathsf{W}_{-1}(\mathbf{D}_j)\|_{\mathcal{B}}$, from (10.19) and (10.21) we conclude that the sequence $(A_n)_{n=1}^\infty$ converges strongly to zero as $n \to \infty$. For $u \in L^p(\mathbb{R})$ from (10.23) it follows that

$$\|u\|_p \le \|Z_{\tau_n}\mathsf{W}_{-1}(\mathbf{B})Z_{\tau_n}^{-1}\|_{\mathcal{B}}\|H_\lambda u\|_p + \|A_n u\|_p \le \|\mathsf{W}_{-1}(\mathbf{B})\|_{\mathcal{B}}\|H_\lambda u\|_p + \|A_n u\|_p.$$

Passing to the limit as $n \to \infty$, we obtain

$$\|u\|_p \le \|\mathsf{W}_{-1}(\mathbf{B})\|_{\mathcal{B}}\|H_\lambda u\|_p$$

for all $u \in L^p(\mathbb{R})$. This means that the kernel of H_λ is trivial and the range of H_λ is closed.

Analogously, from (10.18) it follows that

$$\begin{aligned}&(Z_{\tau_n^*}W_{-1}(\mathbf{B})^* Z_{\tau_n^*}^{-1})H_\lambda^* - I\\ &= Z_{\tau_n^*}K_2^*Z_{\tau_n^*}^{-1} + \sum_{j=1}^l (Z_{\tau_n^*}\mathsf{W}_{-1}(\mathbf{F}_j)Z_{\tau_n^*}^{-1})(Z_{\tau_n^*}W^0(\overline{v_j})Z_{\tau_n^*}^{-1}) =: B_n.\end{aligned}$$

As above, taking into account (10.20) and (10.22), we see that $(B_n)_{n=1}^\infty$ converges strongly to zero on $L^q(\mathbb{R})$ and therefore

$$\|w\|_q \le \|\mathsf{W}_{-1}(\mathbf{B})\|_{\mathcal{B}}\|H_\lambda^* w\|_q$$

for every $w \in L^q(\mathbb{R})$. Hence the kernel of H_λ^* is trivial, too. Thus H_λ is invertible on $L^p(\mathbb{R})$. From Corollary 3.3(c) we conclude that $\lambda \notin \mathfrak{L}_p$. Thus

$$\mathfrak{L}_p \subset \operatorname{sp}_{\mathcal{L}^{\mathcal{J}}_{\infty,\eta}}\left(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P} + \mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q})\right). \tag{10.24}$$

Combining (10.16) and (10.24), we arrive at (10.14).

To prove equality (10.15), we argue similarly, applying Corollary 3.3(b) instead of Corollary 3.3(a) and the homomorphism W_1 instead of W_{-1}. □

10.5. Spectra of the canonical elements in the algebras $\mathcal{A}_\eta^\pm$ and $\mathcal{L}_\eta^\pm$

Now we are ready to calculate the spectra of the canonical elements required in the two idempotents theorem.

Lemma 10.7. *Let $\eta \in M_\infty(SO)$ and $\mathfrak{L}_p$ be the lentiform domain defined by* (3.2).

(a) *If*

$$X_- := p_- r_- p_- + (e_- - p_-)(e_- - r_-)(e_- - p_-),$$

then $\mathrm{sp}_{\mathcal{A}_\eta^-}(X_-) = \mathrm{sp}_{\mathcal{L}_\eta^-}(X_-) = \mathfrak{L}_p$.

(b) *If*

$$X_+ := p_+ r_+ p_+ + (e_+ - p_+)(e_+ - r_+)(e_+ - p_+),$$

then $\mathrm{sp}_{\mathcal{A}_\eta^+}(X_+) = \mathrm{sp}_{\mathcal{L}_\eta^+}(X_+) = \mathfrak{L}_p$.

Proof. (a) Let $\lambda \in \mathbb{C}$. From Lemmas 10.3(a) and 10.2 it follows that the element $\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P} + \mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q} - \lambda\mathbf{I})$ is invertible in $\mathcal{L}_{\infty,\eta}^{\mathcal{J}}$ if and only if

$$\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{X}_-)\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P} + \mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q} - \lambda\mathbf{I})\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{X}_-) = X_- - \lambda e_-$$

is invertible in $\mathcal{L}_\eta^-$ and

$$\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{X}_+)\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P} + \mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q} - \lambda\mathbf{I})\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{X}_+) = -\lambda e_+$$

is invertible in $\mathcal{L}_\eta^+$. Therefore

$$\mathrm{sp}_{\mathcal{L}_{\infty,\eta}^{\mathcal{J}}}\big(\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{P}\mathbf{X}_-\mathbf{W}_-\mathbf{X}_-\mathbf{P} + \mathbf{Q}\mathbf{X}_-\mathbf{W}_+\mathbf{X}_-\mathbf{Q})\big) = \mathrm{sp}_{\mathcal{L}_\eta^-}(X_-) \cup \{0\}.$$

In view of (10.14), the above equality means that $\mathrm{sp}_{\mathcal{L}_\eta^-}(X_-) \cup \{0\} = \mathfrak{L}_p$. But 0 is not an isolated point of $\mathfrak{L}_p$. Due to the compactness of the spectrum $\mathrm{sp}_{\mathcal{L}_\eta^-}(X_-)$ we conclude that $0 \in \mathrm{sp}_{\mathcal{L}_\eta^-}(X_-)$. Hence

$$\mathrm{sp}_{\mathcal{L}_\eta^-}(X_-) = \mathfrak{L}_p. \tag{10.25}$$

Recall that the identity element in the unital algebras $\mathcal{A}_\eta^- \subset \mathcal{L}_\eta^-$ is the same. Since the lentiform domain $\mathfrak{L}_p$ does not separate the complex plane, that is, $\mathfrak{L}_p$ and $\mathbb{C} \setminus \mathfrak{L}_p$ are connected sets, from [22, Corollary of Theorem 10.18] and equality (10.25) it follows that $\mathrm{sp}_{\mathcal{A}_\eta^-}(X_-) = \mathfrak{L}_p$. Part (a) is proved.

The proof of part (b) is analogous. □

10.6. Invertibility in the local algebra $\mathcal{L}_{\infty,\eta}^{\mathcal{J}}$

Now we are in a position to describe the local algebras $\mathcal{A}_{\infty,\eta}^{\mathcal{J}}$ with the aid of the two idempotents theorem.

Theorem 10.8. *Let $\eta \in M_\infty(SO)$ and $\mathfrak{L}_p$ be the lentiform domain defined by* (3.2). *Suppose $x \in \mathfrak{L}_p$.*

(a) *The maps*

$$\begin{aligned}\Sigma_x^-, \Sigma_x^+ :& \big\{\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{P})\big\} \cup \big\{\Phi_{\infty,\eta}^{\mathcal{J}}\big[(aI)\big] : a \in PC\big\} \\ &\cup \big\{\Phi_{\infty,\eta}^{\mathcal{J}}\big[(W^0(b))\big] : b \in [PC_p, SO_p]\big\} \to \mathbb{C}^{2\times 2}\end{aligned}$$

given by

$$\Sigma_x^-(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P})) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \Sigma_x^-(\Phi^{\mathcal{J}}_{\infty,\eta}[(aI)]) := \begin{bmatrix} a(-\infty) & 0 \\ 0 & a(-\infty) \end{bmatrix}, \quad (10.26)$$

$$\Sigma_x^-(\Phi^{\mathcal{J}}_{\infty,\eta}[(W^0(b)]) :=$$

$$\begin{bmatrix} b_\eta(-\infty)x + b_\eta(+\infty)(1-x) & [b_\eta(-\infty) - b_\eta(+\infty)]\sqrt{x(1-x)} \\ [b_\eta(-\infty) - b_\eta(+\infty)]\sqrt{x(1-x)} & b_\eta(-\infty)(1-x) + b_\eta(+\infty)x \end{bmatrix} \quad (10.27)$$

and

$$\Sigma_x^+(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{P})) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \Sigma_x^+(\Phi^{\mathcal{J}}_{\infty,\eta}[(aI)]) := \begin{bmatrix} a(+\infty) & 0 \\ 0 & a(+\infty) \end{bmatrix}, \quad (10.28)$$

$$\Sigma_x^+(\Phi^{\mathcal{J}}_{\infty,\eta}[(W^0(b)]) :=$$

$$\begin{bmatrix} b_\eta(+\infty)x + b_\eta(-\infty)(1-x) & [b_\eta(+\infty) - b_\eta(-\infty)]\sqrt{x(1-x)} \\ [b_\eta(+\infty) - b_\eta(-\infty)]\sqrt{x(1-x)} & b_\eta(+\infty)(1-x) + b_\eta(-\infty)x \end{bmatrix}, \quad (10.29)$$

where $\sqrt{x(1-x)}$ *denotes any complex number such that its square is equal to* $x(1-x)$*, extend to Banach algebra homomorphisms*

$$\Sigma_x^- : \mathcal{A}^{\mathcal{J}}_{\infty,\eta} \to \mathbb{C}^{2\times 2}, \quad \Sigma_x^+ : \mathcal{A}^{\mathcal{J}}_{\infty,\eta} \to \mathbb{C}^{2\times 2}.$$

(b) *Let* $\mathbf{A} \in \mathcal{A}$*. The coset* $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})$ *is invertible in the local algebra* $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$ *if and only if*

$$\det \Sigma_x^-(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})) \neq 0, \quad \det \Sigma_x^+(\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{A})) \neq 0$$

for all $x \in \mathfrak{L}_p$*.*

(c) *The algebra* $\mathcal{A}^{\mathcal{J}}_{\infty,\eta}$ *is inverse closed in the algebra* $\mathcal{L}^{\mathcal{J}}_{\infty,\eta}$*.*

Proof. From Lemma 10.4 we obtain that the algebras $\mathcal{A}^-_\eta$ and $\mathcal{A}^+_\eta$ are subject to the two idempotents theorem. Let $x \in \mathbb{C}$. Define the maps

$$\sigma_x^\pm : \{e_\pm, p_\pm, r_\pm\} \to \mathbb{C}^{2\times 2}$$

by equalities (10.12)–(10.13) with $e_\pm, p_\pm, r_\pm$ and $\sigma_x^\pm$ in place of e, p, r, and σ_x, respectively. By Lemma 10.7, the spectra

$$\mathrm{sp}_{\mathcal{A}^-_\eta}(X_-), \quad \mathrm{sp}_{\mathcal{L}^-_\eta}(X_-), \quad \mathrm{sp}_{\mathcal{A}^+_\eta}(X_+), \quad \mathrm{sp}_{\mathcal{L}^+_\eta}(X_+)$$

all coincide and are equal to the lentiform domain $\mathfrak{L}_p$ given by (3.2). Then, by Theorem 10.5(a), for $x \in \mathfrak{L}_p$ the maps σ_x^- and σ_x^+ extend to Banach algebra homomorphisms

$$\sigma_x^- : \mathcal{A}^-_\eta \to \mathbb{C}^{2\times 2}, \quad \sigma_x^+ : \mathcal{A}^+_\eta \to \mathbb{C}^{2\times 2}.$$

Let $\mathbf{A} \in \mathcal{A}$. From Theorem 10.5(b),(c) we know that the element $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-\mathbf{A}\mathbf{X}_-)$ (respectively, $\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+\mathbf{A}\mathbf{X}_+)$) is invertible in both algebras $\mathcal{A}^-_\eta$ and $\mathcal{L}^-_\eta$ (respectively, in both $\mathcal{A}^+_\eta$ and $\mathcal{L}^+_\eta$) if and only if the element $\sigma_x^-[\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_-\mathbf{A}\mathbf{X}_-)]$ (respectively, $\sigma_x^+[\Phi^{\mathcal{J}}_{\infty,\eta}(\mathbf{X}_+\mathbf{A}\mathbf{X}_+)]$) is invertible in $\mathbb{C}^{2\times 2}$. In particular, the algebra

$\mathcal{A}_\eta^-$ is inverse closed in the algebra $\mathcal{L}_\eta^-$ and the algebra $\mathcal{A}_\eta^+$ is inverse closed in the algebra $\mathcal{L}_\eta^+$.

Further, for $x \in \mathfrak{L}_p$ the maps

$$\Sigma_x^- : \mathcal{A}_{\infty,\eta}^{\mathcal{J}} \to \mathbb{C}^{2\times 2}, \quad \Sigma_x^-[\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A})] = \sigma_x^-[\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{X}_-\mathbf{A}\mathbf{X}_-)],$$
$$\Sigma_x^+ : \mathcal{A}_{\infty,\eta}^{\mathcal{J}} \to \mathbb{C}^{2\times 2}, \quad \Sigma_x^+[\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A})] = \sigma_x^+[\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{X}_+\mathbf{A}\mathbf{X}_+)]$$

are Banach algebra homomorphisms. From (10.12)–(10.13) and (10.5)–(10.10) it follows that Σ_x^- and Σ_x^+ are defined on the generators of $\mathcal{A}_{\infty,\eta}^{\mathcal{J}}$ by formulas (10.26)–(10.29). This finishes the proof of parts (a) and (b).

Taking into account that $\mathcal{A}_\eta^-$ is inverse closed in $\mathcal{L}_\eta^-$ and $\mathcal{A}_\eta^+$ is inverse closed in $\mathcal{L}_\eta^+$, we obtain that $\mathcal{A}_{\infty,\eta}^{\mathcal{J}}$ is inverse closed in $\mathcal{L}_{\infty,\eta}^{\mathcal{J}}$ in view of Lemma 10.3(b). Part (c) is proved. $\square$

Let $R : \mathbb{C} \setminus \big((-\infty,0) \cup (1,+\infty)\big) \to \mathbb{C}$ be a continuous branch of the function $x \mapsto \sqrt{x(1-x)}$. Since $\mathfrak{L}_p \cap \big((-\infty,0)\cup(1,+\infty)\big) = \emptyset$, the function R is continuous on $\mathfrak{L}_p$.

Corollary 10.9. *Let $\eta \in M_\infty(SO)$ and $\mathfrak{L}_p$ be the lentiform domain defined by* (3.2). *The maps*

$$\mathsf{N}_\eta^-, \mathsf{N}_\eta^+ : \{\mathbf{P}\} \cup \big\{(aI) : a \in PC\big\} \cup \big\{(W^0(b)) : b \in [PC_p, SO_p]\big\} \to C(\mathfrak{L}_p, \mathbb{C}^{2\times 2})$$

given for $x \in \mathfrak{L}_p$ by

$$\big(\mathsf{N}_\eta^-(\mathbf{P})\big)(x) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \big(\mathsf{N}_\eta^-[(aI)]\big)(x) := \begin{bmatrix} a(-\infty) & 0 \\ 0 & a(-\infty) \end{bmatrix}, \tag{10.30}$$

$$\big(\mathsf{N}_\eta^-[(W^0(b))]\big)(x) :=$$
$$\begin{bmatrix} b_\eta(-\infty)x + b_\eta(+\infty)(1-x) & [b_\eta(-\infty) - b_\eta(+\infty)]R(x) \\ [b_\eta(-\infty) - b_\eta(+\infty)]R(x) & b_\eta(-\infty)(1-x) + b_\eta(+\infty)x \end{bmatrix} \tag{10.31}$$

and

$$\big(\mathsf{N}_\eta^+(\mathbf{P})\big)(x) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \big(\mathsf{N}_\eta^+[(aI)]\big)(x) := \begin{bmatrix} a(+\infty) & 0 \\ 0 & a(+\infty) \end{bmatrix}, \tag{10.32}$$

$$\big(\mathsf{N}_\eta^+[(W^0(b))]\big)(x) :=$$
$$\begin{bmatrix} b_\eta(+\infty)x + b_\eta(-\infty)(1-x) & [b_\eta(+\infty) - b_\eta(-\infty)]R(x) \\ [b_\eta(+\infty) - b_\eta(-\infty)]R(x) & b_\eta(+\infty)(1-x) + b_\eta(-\infty)x \end{bmatrix}, \tag{10.33}$$

extend to Banach algebra homomorphisms

$$\mathsf{N}_\eta^- : \mathcal{A} \to C(\mathfrak{L}_p, \mathbb{C}^{2\times 2}), \quad \mathsf{N}_\eta^+ : \mathcal{A} \to C(\mathfrak{L}_p, \mathbb{C}^{2\times 2}).$$

Proof. It is clear that N_η^- (resp. N_η^+) is the composition of the canonical Banach algebra homomorphism $\Phi_{\infty,\eta}^{\mathcal{J}}|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_{\infty,\eta}^{\mathcal{J}}$ and the Banach algebra homomorphism Σ_x^- (resp. Σ_x^+) defined in Theorem 10.8(a). Note that the function R was chosen so that $\Sigma_x^{\pm}(\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A}))$ are continuous in $x \in \mathfrak{L}_p$ for $\mathbf{A} \in \mathcal{A}$. □

11. Main result and some corollaries

11.1. Main result

Combining the results of Sections 6–10, we arrive at the main result of the paper.

Theorem 11.1. *Let $\mathfrak{L}_p$ be the lentiform domain defined by* (3.2) *and $\mathcal{A}$ be the algebra from Definition* 4.3. *A sequence $\mathbf{A} \in \mathcal{A}$ is stable if and only if the following three conditions are fulfilled:*

(a) *the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$, and $\mathsf{W}_1(\mathbf{A})$ are invertible in $\mathcal{B}$;*
(b) *the operators $\mathsf{H}_\eta(\mathbf{A})$ are invertible in $\mathcal{B}$ for all $\eta \in \mathbb{R}$;*
(c) *for every $\eta \in M_\infty(SO)$ and every $x \in \mathfrak{L}_p$,*

$$\det\big(\mathsf{N}_\eta^-(\mathbf{A})\big)(x) \neq 0, \quad \det\big(\mathsf{N}_\eta^+(\mathbf{A})\big)(x) \neq 0,$$

where $\mathsf{N}_\eta^{\pm}(\mathbf{A})$ are images of the homomorphisms defined in Corollary 10.9.

Proof. Necessity. If $\mathbf{A} \in \mathcal{A}$ is stable, then condition (b) holds due to Theorem 7.7. Further, in view of Theorem 6.7, condition (a) is fulfilled and the coset $\Phi_{\xi,\eta}^{\mathcal{J}}(\mathbf{A})$ is invertible in the local algebra $\mathcal{L}_{\xi,\eta}^{\mathcal{J}}$ for every pair $(\xi, \eta) \in \Omega$, where Ω is given by (2.8). If $(\xi, \eta) \in \{\infty\} \times M_\infty(SO)$, then applying Theorem 10.8(b) we obtain for all $x \in \mathfrak{L}_p$,

$$\det \Sigma_x^-(\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A})) \neq 0, \quad \det \Sigma_x^+(\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A})) \neq 0.$$

Since $(\mathsf{N}_\eta^-(\mathbf{A}))(x) = \Sigma_x^-(\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A}))$ and $(\mathsf{N}_\eta^+(\mathbf{A}))(x) = \Sigma_x^+(\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A}))$, this immediately implies condition (c). The necessity portion is proved.

Sufficiency. If condition (a) is fulfilled, then from Theorem 8.2 it follows that $\Phi_{\xi,\eta}^{\mathcal{J}}(\mathbf{A})$ is invertible in the local algebra $\mathcal{L}_{\xi,\eta}^{\mathcal{J}}$ with $(\xi, \eta) \in \mathbb{R} \times M_\infty(SO)$. From condition (b) and Theorem 9.3 we infer that the coset $\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A})$ is invertible in the local algebra $\mathcal{L}_{\infty,\eta}^{\mathcal{J}}$ with $\eta \in \mathbb{R}$. From condition (c) in view of Theorem 10.8(b) and Corollary 10.9 it follows that the coset $\Phi_{\infty,\eta}^{\mathcal{J}}(\mathbf{A})$ is invertible in the local algebra $\mathcal{L}_{\infty,\eta}^{\mathcal{J}}$ with $\eta \in M_\infty(SO)$. Thus the coset $\Phi_{\xi,\eta}^{\mathcal{J}}(\mathbf{A})$ is invertible in the local algebra $\mathcal{L}_{\xi,\eta}^{\mathcal{J}}$ for every $(\xi, \eta) \in \Omega$, where Ω is given by (2.8). Combining this fact with condition (a) we see that $\mathbf{A}$ is stable by Theorem 6.7. □

11.2. Applicability of the finite section method

Theorem 11.1 can be specified for the finite section method as follows.

Theorem 11.2. *Suppose $\mathfrak{L}_p$ is a lentiform domain defined by* (3.2). *Let A be an operator in the smallest closed subalgebra of the algebra $\mathcal{B}$ that contains the operators aI with $a \in PC$ and $W^0(b)$ with $b \in [PC_p, SO_p]$. Then the finite section*

method (1.2) *applies to the operator A if and only if the following three conditions hold with $\mathbf{A} = (A)$:*

(a) *the operators*

$$\chi_+ \mathsf{W}_{-1}(\mathbf{A})\chi_+ I + \chi_- I, \quad \chi_- \mathsf{W}_1(\mathbf{A})\chi_- I + \chi_+ I,$$

and A are invertible in $\mathcal{B}$;

(b) *the operators $P_1 \mathsf{H}_\eta(\mathbf{A}) P_1 + Q_1$ are invertible in $\mathcal{B}$ for all $\eta \in \mathbb{R}$;*

(c) *for every $\eta \in M_\infty(SO)$ and every $x \in \mathfrak{L}_p$,*

$$[\mathsf{N}_\eta^-(\mathbf{A})]_{11}(x) \neq 0, \quad [\mathsf{N}_\eta^+(\mathbf{A})]_{11}(x) \neq 0,$$

where $[f]_{11}$ denotes the $(1,1)$-entry of a 2×2 matrix function f.

Proof. From Proposition 5.2(a) it follows that

$$\begin{aligned} \mathsf{W}_{-1}(\mathbf{PAP} + \mathbf{Q}) &= \chi_+ \mathsf{W}_{-1}(\mathbf{A})\chi_+ I + \chi_- I, \\ \mathsf{W}_0(\mathbf{PAP} + \mathbf{Q}) &= A, \\ \mathsf{W}_1(\mathbf{PAP} + \mathbf{Q}) &= \chi_- \mathsf{W}_1(\mathbf{A})\chi_- I + \chi_+ I. \end{aligned}$$

Hence, condition (a) of Theorem 11.1 for the sequence $\mathbf{PAP} + \mathbf{Q}$ is just condition (a) of the present theorem.

From Proposition 7.4(a) we obtain for every $\eta \in \mathbb{R}$,

$$\mathsf{H}_\eta(\mathbf{PAP} + \mathbf{Q}) = P_1 \mathsf{H}_\eta(\mathbf{A}) P_1 + Q_1.$$

This means that condition (b) of Theorem 11.1 specifies to condition (b) of the present theorem for the sequence $\mathbf{PAP} + \mathbf{Q}$.

Finally from (10.30) and (10.32) we get for every $\eta \in M_\infty(SO)$ and $x \in \mathfrak{L}_p$,

$$\begin{aligned} \det\big(\mathsf{N}_\eta^-(\mathbf{PAP} + \mathbf{Q})\big)(x) &= \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{N}_\eta^-(\mathbf{A})(x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} [\mathsf{N}_\eta^-(\mathbf{A})]_{11}(x) & 0 \\ 0 & 1 \end{bmatrix} = [\mathsf{N}_\eta^-(\mathbf{A})]_{11}(x) \qquad (11.1) \end{aligned}$$

and analogously

$$\det\big(\mathsf{N}_\eta^+(\mathbf{PAP} + \mathbf{Q})\big)(x) = [\mathsf{N}_\eta^+(\mathbf{A})]_{11}(x). \qquad (11.2)$$

Equalities (11.1) and (11.2) imply that condition (c) of Theorem 11.1 for the sequence $\mathbf{PAP} + \mathbf{Q}$ coincides with condition (c) of the present theorem. $\square$

11.3. Applicability of the finite section method to paired convolution operators

To illustrate Theorem 11.2, we apply it to the so-called *paired convolution operator*

$$A = W^0(a)\chi_- I + W^0(b)\chi_+ I \quad (a, b \in [PC_p, SO_p]). \qquad (11.3)$$

For $g \in \mathcal{M}_p$, the Wiener-Hopf operator on $L^p(\mathbb{R}_+)$ with symbol g is the operator $W(g)$ defined by

$$W(g) : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+), \quad \varphi \mapsto \chi_+ W^0(g)\chi_+ \varphi.$$

Let J denote the flip operator on the space $L^p(\mathbb{R})$ defined by $(J\varphi)(x) = \varphi(-x)$. For $g \in \mathcal{M}_p$, put $\widetilde{g}(x) := g(-x)$.

Corollary 11.3. *The finite section method* (1.2) *applies to the paired convolution operator given by* (11.3) *if and only if the following three conditions are fulfilled:*

(a) *the operator* (11.3) *is invertible on the space* $L^p(\mathbb{R})$ *and the Wiener-Hopf operators* $W(a)$ *and* $W(\widetilde{b})$ *are invertible on the space* $L^p(\mathbb{R}_+)$*;*

(b) *for every* $\eta \in \mathbb{R}$,

$$a(\eta^\pm) \neq 0, \quad b(\eta^\pm) \neq 0, \quad \operatorname{wind} \mathfrak{C}_p\left(1, \frac{a(\eta^-)}{a(\eta^+)}, \frac{b(\eta^-)}{b(\eta^+)}\right) = 0,$$

where $\operatorname{wind} \mathfrak{C}_p(\cdot,\cdot,\cdot)$ *is defined in Subsection* 3.4*;*

(c) *for every* $\eta \in M_\infty(SO)$ *and every* $x \in \mathfrak{L}_p$,

$$a_\eta(-\infty)x + a_\eta(+\infty)(1-x) \neq 0, \quad b_\eta(+\infty)x + b_\eta(-\infty)(1-x) \neq 0.$$

Proof. Let $\mathbf{A} = (W^0(a)\chi_- I + W^0(b)\chi_+ I)$. Taking into account Proposition 5.2(b)–(c) we see that the operator

$$\chi_+ \mathsf{W}_{-1}(\mathbf{A})\chi_+ I + \chi_- I = \chi_+ W^0(a)\chi_+ I + \chi_- I$$

is invertible on $L^p(\mathbb{R})$ if and only if the Wiener-Hopf operator $W(a)$ is invertible on $L^p(\mathbb{R}_+)$ and that the operator

$$\chi_- \mathsf{W}_1(\mathbf{A})\chi_- I + \chi_+ I = \chi_- W^0(b)\chi_- I + \chi_+ I$$

is invertible on $L^p(\mathbb{R})$ if and only if the operator

$$J\chi_- W^0(b)\chi_- J = \chi_+ J W^0(b) J \chi_+ I = W(\widetilde{b})$$

is invertible on $L^p(\mathbb{R}_+)$. This implies that condition (a) of Theorem 11.2 for the constant sequence $\mathbf{A}$ is nothing but condition (a) of the present corollary.

From Proposition 7.4(b)–(c) it follows that for every $\eta \in \mathbb{R}$,

$$\begin{aligned} P_1 \mathsf{H}_\eta(\mathbf{A}) P_1 + Q_1 &= P_1\big[(a(\eta^-)W^0(\chi_-) + a(\eta^+)W^0(\chi_+))\chi_- I \\ &\quad + (b(\eta^-)W^0(\chi_-) + b(\eta^+)W^0(\chi_+))\chi_+ I\big] P_1 + Q_1 \\ &= \chi_{(-1,1)}\big(a(\eta^-)P_\mathbb{R} + a(\eta^+)Q_\mathbb{R}\big)\chi_{(-1,0)} I \\ &\quad + \chi_{(-1,1)}\big(b(\eta^-)P_\mathbb{R} + b(\eta^+)Q_\mathbb{R}\big)\chi_{(0,1)} I + \chi_{\mathbb{R}\setminus(-1,1)} I. \end{aligned}$$

Obviously this operator is invertible on $L^p(\mathbb{R})$ if and only if the operator

$$P_{(-1,1)}(a(\eta^-)\chi_{(-1,0)} + b(\eta^-)\chi_{(0,1)})I + Q_{(-1,1)}(a(\eta^+)\chi_{(-1,0)} + b(\eta^+)\chi_{(0,1)})I$$

is invertible on $L^p(-1,1)$. By Lemma 3.4, this operator is invertible on $L^p(-1,1)$ if and only if condition (b) is fulfilled. This means that for $\mathbf{A}$ condition (b) of Theorem 11.2 is equivalent to condition (b) of the present corollary.

From formulas (10.30)–(10.33) it follows that

$$\begin{aligned} [\mathsf{N}_\eta^-(\mathbf{A})]_{11}(x) &= a_\eta(-\infty)x + a_\eta(+\infty)(1-x), \\ [\mathsf{N}_\eta^+(\mathbf{A})]_{11}(x) &= b_\eta(+\infty)x + b_\eta(-\infty)(1-x) \end{aligned}$$

for $x \in \mathfrak{L}_p$ and $\eta \in M_\infty(SO)$. This means that condition (c) of Theorem 11.2 specifies to condition (c) of the present corollary. $\square$

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